Lecture 3: 9/11/2017

Reminder:

$H(\vec{p}) = -\sum_i p_i \log(p_i)$

$H_\mu(\xi) = H(\{\mu(A_i) \mid A_i \in \xi\})$

$(X, \mathcal{B}, \mu, T)$ p.p.s

$h_\mu(T, \xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right) = \inf_n \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right)$

$h_\mu(T) = \sup_{H_\mu(\xi)<\infty} h_\mu(T, \xi)$

Let ξ, η be two partitions. $B \in \eta$, we define $\mu|_B(A) = \frac{\mu(A\cap B)}{\mu(B)}$,

$H_\mu(\xi|\eta) = \sum_{B\in\eta} \mu(B) H_{\mu|_B}(\xi)$

2') Let ξ, η, ζ be partitions, then $H_\mu(\xi\vee\eta|\zeta) = H_\mu(\eta|\zeta) + H_\mu(\xi|\zeta\vee\eta)$

Monotonicity:

- $H(\xi\vee\eta|\zeta) \geq H_\mu(\eta|\zeta)$
- $H_\mu(\xi|\eta\vee\zeta) \leq H_\mu(\xi|\eta)$

Continuity bound (last lecture):

$h_\mu(T, \eta) \leq h_\mu(T, \xi) + H_\mu(\eta|\xi)$

Prop.:

$(X, \mathcal{B}, \mu, T)$ p.p.s, ξ a finite entropy partition. Then:

1) For any fixed $k \geq 1$ $h_\mu(T, \xi) = h_\mu\left(T, \bigvee_{i=0}^{k-1} T^{-i}\xi\right)$
2) For T invertible, $h_\mu(T, \xi) = h_\mu(T^{-1}, \xi) = h_\mu\left(T, \bigvee_{i=-k}^{k} T^{-i}\xi\right)$
3) $h_\mu(T^k) = k\, h_\mu(T)$
4) $h_\mu(T) = h_\mu(T^{-1})$ if T is invertible

2) $\Rightarrow$ 4) ← obvious

Proof:

1) $h_\mu(T, \bigvee_{i=0}^{k-1} T^{-i}\xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{j=0}^{n-1} T^{-j}\left(\bigvee_{i=0}^{k} T^{-i}\xi\right)\right) =$

$= \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n+k-1} T^{-i}\xi\right) = \lim_{n\to\infty} \underbrace{\frac{n+k-1}{n}}_{\xrightarrow[n\to\infty]{} 1} \cdot \frac{1}{n+k-1} H_\mu\left(\bigvee_{i=0}^{n+k-1} T^{-i}\xi\right) = h_\mu(T,\xi)$

2) since T is invertible, $T\xi = \{T(A) \mid A\in\xi\} = (T^{-1})^{-1}(\xi)$ is a partition

since μ is T-inv.:

$H_\mu\left(\bigvee_{i=0}^{n-1} T^{i}\xi\right) = H_\mu\left(T^{-(n-1)}\bigvee_{i=0}^{n-1} T^{i}\xi\right) = H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right)$

if we devide by n & take the limit, we will finish:

$h_\mu(T,\xi) = h_\mu(T^{-1},\xi)$. The second equality is similar.

3) $h_\mu(T^k, \bigvee_{i=0}^{k-1} T^{-i}\xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{j=0}^{n-1} T^{-jk}\left(\bigvee_{i=0}^{k-1} T^{-i}\xi\right)\right) =$

$= k \lim_{n\to\infty} \frac{1}{nk} H_\mu\left(\bigvee_{j=0}^{nk-1} T^{-j}\xi\right) = k h_\mu(T,\xi)$.

After taking sup over all ξ, we get: $k h_\mu(T) \le h_\mu(T^k)$.

For the reverse inequality: $h_\mu(T^k, \eta) \le h_\mu(T^k, \bigvee_{i=0}^{k-1} T^{-i}\eta) =$

$= k\, h_\mu(T,\eta)$

by taking the sup of both sides we will finish.

■

Prop.:

For every countable partition ξ there is a finite partition η s.t. $H_\mu(\xi|\eta) < \varepsilon$ & $\eta \subseteq \sigma(\xi)$ - (σ algebra generated by ξ)

In particular: $h_\mu(T) = \sup_{\eta \text{ finite partitions}} h_\mu(T,\eta)$

Proof:

First we explain why the first assertion implies

$$\sup_{H_\mu(\xi)<\infty} h_\mu(T,\xi) = \sup_{\eta \text{ finite}} h_\mu(T,\eta).$$

$\ge$ - obvious

for $\le$, let $\varepsilon > 0$, let η as in first assertion.

⇓

by continuity $h_\mu(T,\xi) \le h_\mu(T,\eta) + H_\mu(\xi|\eta) < \sup_{\eta \text{ finite partition}} h_\mu(T,\eta) + \varepsilon$

we now prove first assertion. if ξ is finite, nothing to prove. write $\xi = \{A_1, A_2, A_3, \dots\}$. Given N, define:

$B_N = X \setminus \bigcup_{i=1}^{N} A_i$, $\eta = \eta_N = \{A_1, \dots, A_N, B_N\}$. clearly $\eta \subseteq \sigma(\xi)$

& $\mu(B_N) \xrightarrow[N\to\infty]{} 0$ since $\mu(X) = 1$

$$H_\mu(\xi|\eta) = \underbrace{\sum_{i=1}^{N} \mu(A_i) H_{\mu|_{A_i}}(\xi)}_{(*)} + \mu(B_N) H_{\mu|_{B_N}}(\xi) =$$

$$= \mu(B_N) \sum_{j=1}^{\infty} \frac{\mu(A_j)}{\mu(B_N)} \log\left(\frac{\mu(A_j)}{\mu(B_N)}\right) = -\sum_{j=N+1}^{\infty} \mu(A_j) \log\left(\frac{\mu(A_j)}{\mu(B_N)}\right) =$$

$$= \underbrace{-\sum_{j=N+1}^{\infty} \mu(A_j) \log(\mu(A_j))}_{\text{tail of } H_\mu(\xi) \ \xrightarrow[N\to\infty]{} 0} + \underbrace{\sum_{j=N+1}^{\infty} \mu(A_j) \log(\mu(B_N))}_{= \mu(B_N)\cdot\log(\mu(B_N))}$$

(*) is zero for all i, since $\mu|_{A_i}(A_j) = \delta_{ij}$

∎

Consequence of (a'):

$$H_\mu(\zeta \vee \eta \mid \xi) \le H_\mu(\eta \mid \xi) + H_\mu(\zeta \mid \eta)$$

If ζ refines η then:

$$H_\mu(\zeta \mid \xi) \le H_\mu(\zeta \mid \eta) + H_\mu(\eta \mid \xi)$$

<u>Thm:</u> (entropy of a factor)

If $(Y, \mathcal{B}_Y, \nu, S)$ is a p.p.s, which is a factor of $(X, \mathcal{B}, \mu, T)$ then $h_\nu(S) \le h_\mu(T)$. In particular, entropy is an invariant of p.p.s's (preserved under isomorphisms)

<u>Proof:</u>

By definition of a factor $\exists\, X_0 \subseteq X$ ($X_0 \in \mathcal{B}$, $T^{-1}(X_0) = X_0$) conull (i.e. $\mu(X \setminus X_0) = 0$)
& $F : X_0 \longrightarrow Y$ which is measurable $(F^{-1}(\mathcal{B}_Y) \subset \mathcal{B}_X)$
s.t. $F_*\mu = \nu$, $S \circ F = F \circ T$ & :

$$\begin{array}{ccc} X \supset X_0 & \xrightarrow{T} & X_0 \\ F \downarrow & & \downarrow F \\ Y & \xrightarrow[S]{} & Y \end{array}$$

is commutative.

Let ξ be a partition of Y, then $F^{-1}(\xi) = \{F^{-1}(A) \mid A \in \xi\}$ is a partition of X_0.

$$H_\mu(F^{-1}(\xi)) = H_\nu(\xi)$$

$\uparrow$

$F_*\mu = \nu$ means: $\forall A \in \mathcal{B}_Y \quad \nu(A) = \mu(F^{-1}(A))$

Since $S \circ F = F \circ T$, for any n, $F^{-1}\left(\bigvee_{i=0}^{n-1} S^{-i}(\xi)\right) = \bigvee_{i=0}^{n-1} T^{-i}(F^{-1}(\xi))$

so for each n, $H_\nu\left(\bigvee_{i=0}^{n-1} S^{-i}(\xi)\right) = H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}(F^{-1}(\xi))\right)$

So $h_\nu(S, \xi) = h_\mu(T, F^{-1}(\xi))$. Taking sup over ξ, $H_\nu(\xi) < \infty$ then $h_\nu(S) \le h_\mu(T)$

Historical background:

Kolmogorov & Sinai introduced entropy as an invariant of p.p.s's in the 1950's, showed how to compute in examples (we will do this) making it possible to show that some p.p.s's are not isomorphic.

D. Orenstein proved a famous converse (Ornstein isomorphism theorem) in 1970's. $(X, \mathcal{B}, \mu, T)$ is called a Bernoulli System if it is isomorphic to $(A^{\mathbb{Z}}, \text{Borel}, \mu^{\otimes \mathbb{Z}}, \sigma)$ where A is finite or countable, μ is a prob on A, Borel w.r.t Tychonov top., σ left shift.

Ornstein showed: Bernoulli systems with same entropy are isomorphic.

Recall: If $(X, \mathcal{B}, \mu)$ is a p.p.s then the completion of $\mathcal{B}$ with respect to μ is $\bar{\mathcal{B}} = \sigma(\mathcal{B} \cup \{A, cX \mid \exists A \in \mathcal{B}, A \subset A, \mu(A) = 0\})$

μ extends to the completion.

For any $B \in \bar{\mathcal{B}}$ $\exists A \in \mathcal{B}$ s.t $\mu(A \triangle B) = 0$. Given ξ a partition in $\bar{\mathcal{B}}$, can form η a partition in $\mathcal{B}$, s.t

$\forall B \in \xi$ $\exists A = A(B) \in \eta$ s.t $\bar{\mu}(A \triangle B) = 0$.

Then $H_{\mu}(\xi) = H_{\mu}(\eta)$.

If T is measurable w.r.t $\mathcal{B}$, then also w.r.t $\bar{\mathcal{B}}$, & for any n, $H_{\mu}\left(\bigvee_{i=0}^{n-1} T^{-i}(\xi)\right) = H_{\mu}\left(\bigvee_{i=0}^{n-1} T^{-i}(\eta)\right)$

The entropy of T (seen using $(X, \mathcal{B}, \mu, T)$) will not change (be the same as the entropy seen using $(X, \bar{\mathcal{B}}, \bar{\mu}, T)$).

Def:

$(X, \mathcal{B}, \mu, T)$ a p.p.s, ξ a partition. we say that ξ is a <u>one-sided generator</u> if $\bigvee_{n=0}^{\infty} T^{-n}(\xi) = \mathcal{B}$.

T is <u>invertible</u>, we say ξ is a <u>generator</u> if

$$\bigvee_{n=-\infty}^{\infty} T^{-n}(\xi) = \mathcal{B}$$

<u>Thm:</u> (Kolmogorov - Sinai)

If ξ is a one-sided generator of finite entropy (or T is invertible & ξ is a generator) then $h_\mu(T) = h_\mu(T, \xi)$.

i.e. $h_\mu(T, \xi) = \max_{H_\mu(\eta) < \infty} h_\mu(T, \eta)$

<u>Notation:</u> $\bigvee_{i=0}^{\infty} \xi_i = \sigma\left(\bigcup \bigvee_{i=1}^{k} \xi_i\right)$

<u>Remark:</u> $\bigvee_{n=0}^{\infty} T^{-n}(\xi)$ is countably generated!

<u>Lemma:</u> (Continuity)

Let $(X, \mathcal{B}, \mu, T)$ be a p.p.s, ξ a one sided generator of finite entropy, η any finite entropy partition.

Then $H_\mu\left(\eta \mid \bigvee_{i=0}^{n-1} T^{-i}\xi\right) \xrightarrow[n\to\infty]{} 0$

<u>Proof of Kolmogorov - Sinai assuming lemma:</u>

let ξ be a one sided generator, η any finite entropy partition. Then: $h_\mu(T, \eta) \leq h_\mu\left(T, \bigvee_{i=0}^{n-1} T^{-i}\xi\right) + H_\mu\left(\eta \mid \bigvee_{i=0}^{n-1} T^{-i}\xi\right)$

continuity prop from last week. ($\leq$)

$h_\mu\left(T, \bigvee_{i=0}^{n-1} T^{-i}\xi\right) = h_\mu(T, \xi)$

continuity lemma $\to 0$

The case of a generator is similar but continuity lemma should be proved for $\bigvee_{i=-n}^{n} T^{-i}\xi$.

Proof of lemma:

Note: suffices to prove when η is finite.

Because: by Prop. (finite entropy vs. finite partition). $\forall \varepsilon > 0$ $\exists$ finite partition η' s.t $H_\mu(\eta|\eta') < \varepsilon$ & $\eta' \subseteq \sigma(\eta)$

$$H_\mu\left(\eta \vee \eta' \,\Big|\, \bigvee_{i=0}^{n-1} T^{-i}\xi\right) \le H_\mu\left(\eta' \,\Big|\, \bigvee_{i=0}^{n-1} T^{-i}\xi\right) + H_\mu(\eta|\eta')$$

$$\searrow 0 \qquad\qquad \wedge\ \varepsilon$$

so $\limsup_{n\to\infty} H_\mu\left(\eta \,\Big|\, \bigvee_{i=0}^{n-1} T^{-i}\xi\right) \le \varepsilon$. $\varepsilon > 0$ was arbitrary, so done.

So we continue, assume η is finite. $\eta = \{B_1, \dots, B_m\}$

to explain the Idea, suppose that for each i there is n s.t. $B_i \in \sigma\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right)$. for n large enough, $\eta \subseteq \sigma\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right)$, for any $A \in \bigvee_{i=0}^{n-1} T^{-i}\xi$, $\mu|_A$ (element of η) is a vector of 0's & 1's. So $H_\mu\left(\eta \,\Big|\, \bigvee_{i=0}^{n-1} T^{-i}\xi\right) = 0$.

Claim:

For any $A \in \mathcal{B}$ & any $\delta > 0$, $\exists\, n, B \in \sigma\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right)$, s.t. $\mu(A \triangle B) < \delta$.

Define $\mathcal{B}_0 = \left\{ A \in \mathcal{B} \,\Big|\, \forall \delta.\ \exists n, \exists B \in \sigma\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right) \text{ s.t } \mu(A \triangle B) < \delta \right\}$

One can check that $\mathcal{B}_0$ is a σ-algebra. So, (since ξ is a one sided generator) $\mathcal{B}_0 = \mathcal{B}$.

Fix $\varepsilon > 0$. we are going to choose δ (depending on ε & on $B_1, \dots, B_n$) below. By claim, & finiteness of η, there is n & $A_i \in \sigma\left(\bigvee_{j=0}^{n-1} T^{-j}(\xi)\right)$, $i = 1, \dots, n$ s.t. for each i, $\mu(A_i \triangle B_i) < \frac{\delta}{m^2}$.

Define $A_1' = A_1$, $A_2' = A_2 \setminus A_1'$, ..., $A_j' = A_j \setminus \bigcup_{k<j} A_k'$, ... $A_m' = X \setminus \bigcup_{j<m} A_j'$. $\{A_1', \dots, A_m'\}$ is a partition. If $j \le m-1$, then $\mu(A_j' \triangle B_j) = \mu(A_j' \setminus B_j) + \mu(B_j \setminus A_j') \le$

$\le \mu(A_j \setminus B_j) + \mu(B_j \setminus A_j) + \mu\left(B_j \cap \bigcup_{k<j} A_k'\right) \le \mu(A_j \triangle B_j) + \sum_{k \le j} \mu(A_k | B_k)$

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$\overset{(5)}{\leq} \frac{\delta}{m^2} + \sum_{k<j} \frac{\delta}{m^2} \leq \frac{\delta}{m}$.

One can check, using $B_m = X \setminus \bigcup B_i$ & $A'_m = X \setminus \bigcup A'_i$

that $\mu(A'_m \triangle B_m) \leq \delta$ (exercise).

Let $\zeta = \{A'_1, \dots, A'_m\}$. η, ζ are partitions & for each i,

$\mu(B_j \triangle A'_j) < \delta$. $\zeta \subset \sigma\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right)$

$H_\mu\left(\eta \mid \bigvee_{i=0}^{n-1} T^{-i}\xi\right) \leq H_\mu(\eta \mid \zeta)$ ← this can be made ε-close to 0 by taking δ small & using cont. of $H(p_1, \dots, p_m) = -\sum_{i=1}^{m} p_i \log p_i$

$< \varepsilon$

choose δ small enough so that $H(p_1, \dots, p_m) < \varepsilon$ whenever p_i are of the form $\mu|_{A_j}(\eta)$, $\mu(A'_i \triangle B_i) < \delta$