Lecture 6: 25/4/2018

Background on the Fourier transform.

The Schwarts space:

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^{\infty}(\mathbb{R}) \text{ s.t } \forall \alpha, \beta \geq 0 \text{ integers} \quad \sup_{x \in \mathbb{R}} \left| x^{\alpha} \frac{d^{\beta} f}{dx^{\beta}} \right| < \infty \right\}$$

i.e: $|f^{(\beta)}(x)| << \frac{1}{1+|x|^{\alpha}}$.

Exercise:

$f(x) = e^{-\pi x^2}$, $f'(x) = -2\pi x e^{-\pi x^2}$ are all << from polynomial

$f^{(n)}(x) = P_n(x) e^{-\pi x^2}$

For $f \in \mathcal{S}$ the Fourier transform is $\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y} dx$

Lemma:

$$F: \mathcal{S} \longrightarrow \mathcal{S}$$
$$f \longmapsto \hat{f}$$

i.e. $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$

Properties:

1) $\frac{d}{dx} \hat{f} = \pm \widehat{2\pi i x f}$

2) F interwines translation $\left(T_z f(x) := f(x+z)\right)$ & multiplication by $e^{2\pi i z}$:

$$F(T_z f) = e^{2\pi i z} F(f)(x)$$

Convolution: $(f * g)(x) = \int_{\mathbb{R}} f(y) g(x-y) dy$ then $\widehat{f * g} = \hat{f} \cdot \hat{g}$

Dilation:

$D_\lambda f(x) := f(\frac{x}{\lambda})$ then $\widehat{D_\lambda f}(y) = \lambda \hat{f}(\lambda y)$

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$$

Ex (Euler theorem):

$f(x) = e^{-\pi x^2}$ then: $\hat{f} = f$

Ex

$f = \mathbb{1}_{[-a,a]} \notin \mathcal{S}$, $\hat{f}(x) = \dfrac{\sin(\pi a x)}{\pi x}$

Thm.:

$F : \mathcal{S} \to \mathcal{S}$ extends to an isometry of $L^2(\mathbb{R})$ &

$\|\hat{f}\|_2 = \|f\|_2$, $\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 dx$

Fourier inversion:

$f \in \mathcal{S}$ $\hat{\hat{f}} = f(-x)$ i.e. $f(x) = \int \hat{f}(y) e^{2\pi i x y} dy$

Riemann-Lebesgue Lemma:

If $f \in L^1(\mathbb{R})$ then $\hat{f}(y) \xrightarrow{|y| \to \infty} 0$

Poisson Summation formula:

$f \in \mathcal{S}$ then $\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m)$

Multi-dimension:

$$\mathcal{S}(\mathbb{R}^d) = \{ f \in C^\infty(\mathbb{R}^d) \mid \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty \}$$

$\beta, \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$

$\partial^\beta f = \dfrac{\partial^{\beta_1 + \dots + \beta_d} f}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}}$

$$\hat{f}(x) := \int_{\mathbb{R}^d} f(y) e^{-2\pi i x y} dy$$

Same results hold, in particular

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{m \in \mathbb{Z}^d} \hat{f}(m)$$

Goal:

$$N(R) := \#\{\mathbb{Z}^2 \cap B(0,R)\} = \#\{n \in \mathbb{Z}^2 \mid |n| \le R\}$$

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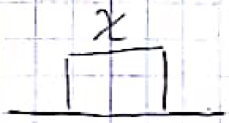
then $N(R) = \pi R^2 + O(R^{2/3})$ (we saw $O(R)$)

Conj.:

$$O(R^{\frac{1}{2}+\varepsilon})$$

Want to use Poisson summation

$$\chi(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases} = \mathbb{1}_{B(0,1)}(x)$$



$$N(R) = \sum_{n \in \mathbb{Z}^2} \chi\left(\tfrac{n}{R}\right).$$ Problem: $\chi \notin \mathcal{S}(\mathbb{R})$

If we could, then get $N(R) = \sum_{m \in \mathbb{Z}^2} \hat{\chi}_R(m)$

$\chi_R(x) = \chi\left(\frac{x}{R}\right)$ & $\hat{\chi}_R(y) = R^2 \hat{\chi}(Ry)$

The "zero-mode" $m = 0$ gives

$$R^2 \hat{\chi}(0) = R^2 \int_{\mathbb{R}^2} \chi(x)\,dx = \pi R^2$$

$\uparrow$ area $\{|x| \le 1\}$

$$(*) \quad N(R) \text{ "="} R^2 \sum_{m \in \mathbb{Z}^2} \hat{\chi}(Rm)$$

Notice:

$$N(R) = \pi R^2 + R^2 \sum_{m \ne 0} \hat{\chi}(Rm)$$

Know: $\hat{\chi}(Rm) \xrightarrow{R \to \infty} 0$ if $m \ne \vec{0}$ (Riemann-Lebesgue)

Last time we saw $\hat{\chi}(y) \ll \frac{1}{|y|^{3/2}}$, $|y| \longrightarrow \infty$

(& we stated that $\hat{\chi}(y) \sim \frac{\cos(|y| - \frac{\pi}{4})}{|y|^{3/2}}$)

This will give $\sum_{0 \ne m \in \mathbb{Z}^2} R^2 \hat{\chi}(Rm) \ll R^2 \sum_{m \ne 0} \frac{1}{(R|m|)^{3/2}}$ &

$$N(R) - \pi R^2 \text{ "}\ll\text{" } R^{1/2} \sum_{0 \ne m \in \mathbb{Z}^2} \frac{1}{|m|^{3/2}}$$

Problem:

$$\sum_{0\neq m\in\mathbb{Z}^2}' \frac{1}{|m|^{3/2}} = \infty \quad \text{Compare with integral}$$

$$\int_{|m|\geq 1} \frac{1}{|m|^{3/2}}\,dm = \int_{\theta=0}^{2\pi}\int_{r=1}^{\infty} \frac{r\,dr\,d\theta}{r^{3/2}} = 2\pi\int_1^\infty \frac{dr}{r^{1/2}} = \infty \quad \text{Fail!!}$$

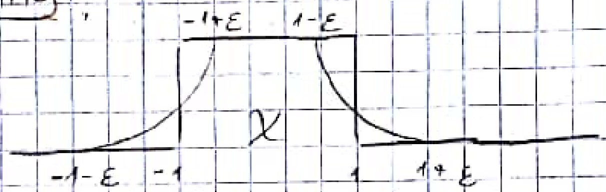
2nd attempt:

1) replace $N(R)$ by a smooth count $N_\varepsilon(R)$

2) Prove an optimal result $N_\varepsilon(R) = \pi R^2 + O\left(\frac{1}{\varepsilon^2}\right)$

3) Go back to $N(R)$

Smooth counting



The new function will be χ_ε. Let $\psi(t)\in C_c^\infty(\mathbb{R})$ be a "bump function" s.t.:

1) $\psi(t) = \psi(-t)$

2) $0\leq\psi\leq 1$

3) Supported in $[-1,1]$

4) $\int_0^\infty \psi(t)\,|t|\,dt = \frac{1}{2\pi}$

For $\varepsilon>0$ define $\overline{\Psi}_\varepsilon(\vec{x})$ by $\overline{\Psi}_\varepsilon(\vec{x}) = \frac{1}{\varepsilon^2}\psi\left(\frac{|\vec{x}|}{\varepsilon}\right) = \begin{cases} \\ 0 & |\vec{x}|>\varepsilon\end{cases}$

& $$\int_{\mathbb{R}^2}\overline{\Psi}_\varepsilon(x)\,dx = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}\frac{1}{\varepsilon^2}\psi\left(\frac{r}{\varepsilon}\right)r\,dr\,d\theta =$$

$$= 2\pi\frac{1}{\varepsilon^2}\int_0^\infty \psi\left(\frac{r}{\varepsilon}\right) r\,dr = 2\pi\int_0^\infty u\,\psi(u)\,du = 1$$

$r = \varepsilon u,\ r\,dr = \varepsilon^2 u\,du$

Define $\chi_\varepsilon(x) = \chi * \overline{\Psi}_\varepsilon(x)$, $\chi(x) = \begin{cases} 1 & |x|\leq 1 \\ 0 & |x|>1\end{cases}$

Why is Poisson summation true?

Need that if $F \in C^\infty(\mathbb{R}/\mathbb{Z})$ then $F(x+1) = F(x)$

F's fourier series $F(x) = \sum_{n\in\mathbb{Z}} \hat{F}(n) e^{2\pi i n x}$

Now take $f \in \mathcal{S}(\mathbb{R})$, want $\sum_{\mathbb{Z}} f(n) = \sum_{\mathbb{Z}} \hat{f}(m)$

Define $F_f(x) = \sum_{n\in\mathbb{Z}} f(x+n)$; $F_f(x+1) = F_f(x)$

$F_f \in C^\infty(\mathbb{R}/\mathbb{Z})$. Use Fourier series of F_f

$$\hat{F}_f(m) = \langle F_f, e^{2\pi i m x} \rangle_{L^2(\mathbb{R}/\mathbb{Z})} = \int_0^1 F_f(x) e^{-2\pi i m x}\, dx =$$

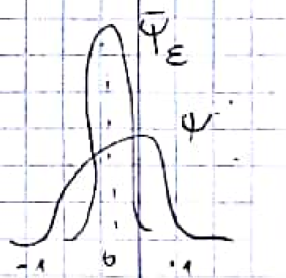
$$= \int_0^1 \sum_n f(x+n) e^{-2\pi i m x} dx = \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n) e^{-2\pi i m (x+n)} dx =$$

$$= \int_{-\infty}^{\infty} f(y) e^{-2\pi i m y} dy = \hat{f}(m)$$

$$\Longrightarrow \hat{F}_f(m) = \hat{f}(m)$$

$$\forall x. \sum_{n\in\mathbb{Z}} f(x+n) = F_f(x) = \sum \hat{F}_f(m) e^{2\pi i m x} = \sum_m \hat{f}(m) e^{2\pi i m x}$$

take $x = 0$, then $\sum_{\mathbb{Z}} f(n) = \sum_{\mathbb{Z}} \hat{f}(m)$

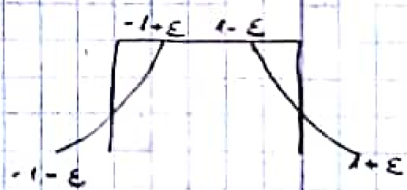


Now $\chi(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$, $\chi_\varepsilon = \chi * \Phi_\varepsilon$

& $\Phi_\varepsilon(\vec{x}) = \frac{1}{\varepsilon^2} \Psi\left(\frac{\vec{x}}{\varepsilon}\right)$

Claim:

$$\chi_\varepsilon(y) = \begin{cases} 1 & |y| < 1-\varepsilon \\ 0 & |y| > 1+\varepsilon \end{cases}$$



Lets check if $|y| < 1-\varepsilon$

$$\chi_\varepsilon(y) = (\chi * \Phi_\varepsilon)(y) = \int_{\mathbb{R}^2} \chi(t) \Phi_\varepsilon(y-t)\, dt =$$

$$= \int_{\mathbb{R}^2} \chi(y-t) \Phi(t)\, dt$$

$|t| < \varepsilon$

Now $|y-t| \leq 1-\varepsilon+\varepsilon = 1$ so $\chi(y-t)=1$ i.e

$$\chi_\varepsilon(y) = \int_{\mathbb{R}^2} \chi(y-t)\overline{\Psi}_\varepsilon(t)\,dt = \int_{\mathbb{R}^2} \overline{\Psi}_\varepsilon(t)\,dt = \int_{\mathbb{R}} \Psi_1(t)\,dt = 1$$

Ex:

If $|y| > 1+\varepsilon$ $\chi_\varepsilon(y) = 0$

Now define $N_\varepsilon(R) = \sum_{n\in\mathbb{Z}^2}' \chi_\varepsilon(\frac{n}{R})$. (Compare $N(R) = \sum_{n\in\mathbb{Z}^2}' \chi(\frac{n}{R})$)

Lemma:

$$N(R(1-\varepsilon)) < N_\varepsilon(R) < N(R(1+\varepsilon))$$

proof:

$\mathbb{1}_{B(0,1-\varepsilon)} \leq \chi_\varepsilon(y) < \mathbb{1}_{B(0,1+\varepsilon)}$ So

$$N_\varepsilon(R) = \sum' \chi_\varepsilon(\tfrac{n}{R}) \leq \sum' \mathbb{1}_{B(0,1+\varepsilon)}(\tfrac{n}{R}) = \sum_{n\in\mathbb{Z}^2}' \chi\left(\tfrac{n}{(1+\varepsilon)R}\right) =: N((1+\varepsilon)R)$$

Corollary:

$$N_\varepsilon\left(\frac{R}{1+\varepsilon}\right) < N(R) < N_\varepsilon\left(\frac{R}{1-\varepsilon}\right)$$

Prop:

$$N_\varepsilon(R) = \pi R^2 + O\left(\frac{1}{\varepsilon^{1/2}}\right) \quad \forall \varepsilon>0, R>0$$

Assuming this, lets show $N(R) = \pi R^2 + O(R^{2/3})$

$$\text{Cor} \implies N(R) \leq N_\varepsilon\left(\frac{R}{1-\varepsilon}\right) \overset{\text{prop.}}{=} \pi\left(\frac{R}{1-\varepsilon}\right)^2 + O\left(\frac{1}{\varepsilon^{1/2}}\right)$$

$$\text{IV} \quad N_\varepsilon\left(\frac{R}{1+\varepsilon}\right) = \pi\left(\frac{R}{1+\varepsilon}\right)^2 + O\left(\frac{1}{\varepsilon^{1/2}}\right)$$

$$\left(\frac{R}{1-\varepsilon}\right)^2 = R^2(1+O(\varepsilon)) = R^2 + O(\varepsilon R^2)$$

$$\pi R^2 + O(\varepsilon R^2) + O(\varepsilon^{-1/2}) \leq N(R) \leq \pi R^2 + O(\varepsilon R^2) + O\left(\frac{1}{\varepsilon^{1/2}}\right)$$

So:

$$|N(R) - \pi R^2| \ll \varepsilon R^2 + \frac{1}{\varepsilon^{1/2}}$$

Minimizeing right side we want $\varepsilon R^2 = \frac{1}{\varepsilon^{1/2}} \Rightarrow \varepsilon = R^{-4/3}$

& we get

$$N(R) = \pi R^2 + O(R^{2/3})$$

$\boxtimes$

Now we will want to prove the Prop. to finish.

Notice $\chi_\varepsilon \in C_c^\infty(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)$

Let $f_R(x) = \chi_\varepsilon(\frac{x}{R}) \in \mathcal{S}(\mathbb{R}^2)$

$N_\varepsilon(R) = \sum_{n\in\mathbb{Z}^2} f_R(n)$. can apply Poisson Summation,

$N_\varepsilon(R) = \sum_{m\in\mathbb{Z}^2} \hat f_R(m)$, $\hat f_R(y) = R^2 \hat\chi_\varepsilon(Ry) = R^2\hat\chi(Ry)\hat\varphi_\varepsilon(Ry)$

<u>Note:</u> in $\mathbb{R}^d$ $f_R(x) = f(\frac{x}{R})$, $\hat f_R(x) = R^d \hat f(Rx)$

Notice $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^2}\varphi(\frac{x}{\varepsilon})$, $\hat\varphi_\varepsilon(z) = \hat\varphi(\varepsilon z)$

So $\hat f_R(y) = R^2\hat\chi(Ry)\hat\varphi(\varepsilon Ry)$

$$N_\varepsilon(R) = \sum_{m\in\mathbb{Z}^2} R^2\hat\chi(Rm)\hat\varphi(\varepsilon Rm) = R^2\underbrace{\hat\chi(0)}_{=\pi}\underbrace{\hat\varphi(0)}_{=1} + \underbrace{\sum_{0\neq m\in\mathbb{Z}^2}\text{same}}$$

We will want to prove this $O(\frac{1}{\varepsilon^{1/2}})$

$$\hat\chi_0(0) = \int \chi(x)\,dx = \text{area}(|x|\le 1) = \pi$$

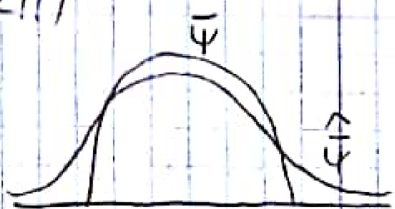
<u>Want:</u>

$$R^2 \sum_{0\neq m\in\mathbb{Z}^2} \hat\chi(Rm)\hat\varphi(\varepsilon Rm) \ll \frac{1}{\varepsilon^{1/2}}$$

Pretend $\hat\varphi$ is also compactly supported by $[-100, 100]$

$$\Rightarrow \text{Error} \ll R^2 \sum_{\substack{0\neq m\in\mathbb{Z}^2 \\ \varepsilon R|m|<100}} \hat\chi(Rm) \ll R^2 \sum_{0<|m|<\frac{100}{\varepsilon R}} \frac{1}{(R|m|)^{3/2}} =$$

$$= R^{\frac{1}{2}} \sum_{\substack{0<|m|<M \\ M=\frac{100}{\varepsilon R}}} \frac{1}{|m|^{3/2}} \quad \text{(home work!!)}$$



So Error $\ll R^{1/2}\left(\frac{1}{\varepsilon R}\right)^{\frac{1}{2}} = \frac{1}{\varepsilon^{1/2}}$

$\boxtimes$

Problem:

If $\psi \neq 0$ is compactly supported then $\hat{\psi}$ cannot be compactly supported.

$$\hat{\psi}(z) = \int_{-1}^{1} \psi(y) \underbrace{e^{-2\pi i y z}}_{\text{analytic in } z} dy$$

If $\psi \in C_c^\infty(\mathbb{R})$ then $\hat{\psi}(z)$ extends to an entire function in $z \in \mathbb{C}$. So cannot vanish on an interval.

So repeat argument without truncating at $|m| < \frac{1}{\varepsilon R}$

$$\sum_{0 \neq m \in \mathbb{Z}^2} R^2 \hat{\chi}(Rm) \hat{\hat{\psi}}(\varepsilon R m) \ll \sum_{0 \neq m} R^2 \frac{1}{(R|m|)^{3/2}} \hat{\hat{\psi}}(\varepsilon R m) =$$

$$= R^{1/2} \sum_{m \neq 0} \frac{1}{|m|^{3/2}} \hat{\hat{\psi}}(\varepsilon R m) \ll R^{1/2} \int_{|x| \geq 1} \frac{1}{|x|^{3/2}} \hat{\hat{\psi}}(\varepsilon R m)\, dx =$$

$\varepsilon R x = y, \quad dy = (\varepsilon R)^2 dx$

$$\overset{\downarrow}{=} R^{\frac{1}{2}} \int \frac{1}{|x|^{3/2}} \hat{\psi}(\varepsilon R x)\, dx = R^{1/2} \int_{|y| > \varepsilon R} \hat{\hat{\psi}}(y) \frac{dy}{(\varepsilon R)^2} \frac{(\varepsilon R)^{3/2}}{|y|^{3/2}} \underset{\uparrow}{\leq}$$

[illegible]
[illegible]

$$\leq \frac{1}{\varepsilon^{1/2}} \int_{y < R^2} \frac{1}{|y|^{3/2}} \hat{\psi}(y)\, dy = \frac{1}{\varepsilon^2} \underbrace{\int_0^\infty \frac{1}{r^{3/2}} \hat{\hat{\psi}}(r, 0)\, r\, dr}_{\text{decays rapidly}}\, 2\pi \ll \frac{1}{\sqrt{\varepsilon}}$$

$$\int_0^\infty \frac{1}{r^{1/2}} \underbrace{\hat{\hat{\psi}}(r, 0)}_{\theta(r)} dr < \infty$$

So we didn't cheat.

Next time:

Thm:

$|N(R) - \pi R^2| > R^{1/2}$ (Known $\log(R)^{1/4}$) for arbitrary large R.

Cramer 1920: $\frac{1}{R} \int_1^R \left| \frac{N(r) - \pi r^2}{\sqrt{r}} \right|^2 dr \overset{R \to \infty}{\sim} C > 0$

Kai Man Tsang 1990: $\lim_{R \to \infty} \frac{1}{R} \int_1^R \left(\frac{N(r) - \pi r^2}{\sqrt{r}} \right)^3 dr = C_3 \neq 0$