2/11/14. Prob. Methods in Comb.

W2D1. (Missed from last lecture).

A family $\mathcal{F}$ of $P[n]$ intersecting iff $\forall A,B \in \mathcal{F}$, $A \cap B \neq \emptyset$.

How large can an intersecting $\mathcal{F}$ be.

Lower bound - $\mathcal{F}_0 = \{A \in P[n] \mid 1 \in A\}$, $|\mathcal{F}| = 2^{n-1}$.

Upper " - We take at most one from A, A^c so

$$|\mathcal{F}| \leq \frac{|P[n]|}{2} = 2^{n-1}.$$

Now, for $\mathcal{F} \in \binom{[n]}{k}$ ($\forall A \in \mathcal{F}$, $|A| = k$).

The trivial $\mathcal{F}_0 \cap \binom{[n]}{k}$ is of size $\binom{n-1}{k-1}$. If $2k > n$

then $\mathcal{F} = \binom{[n]}{k}$ works better. What if $2k \leq n$?

Thm (Erdős, Rogo, Ko) if $k \leq \frac{n}{2}$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting

then $|\mathcal{F}| \leq \binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$

Proof (Katova, 72) Choose $\underbrace{\sigma \in S_n}_{\text{and } i \in \mathbb{Z}/n \text{ uni. ind.}}$ and replace $[n]$ by $\mathbb{Z}/n\mathbb{Z}$.

$A_i = A_i(\sigma) = \{\sigma(i), \sigma(i+1), \dots, \sigma(i+k-1)\}$ - uni. selected in $\binom{[n]}{k}$

$\mathbb{P}(A_i \in \mathcal{F}) = \frac{|\mathcal{F}|}{\binom{n}{k}}$. We want an upper bound on $\mathbb{P}$.

Fix σ. We claim $\mathbb{P}(A_i \in \mathcal{F} \mid \sigma) \leq \frac{k}{n}$.

Assume it's >0. Then there's some i s.t.

$A_i \in \mathcal{F}$. Then $\mathcal{F}$ can't contain $2k-2$ other A_j's

and they can be paired as A_{i-j}, A_{i-j+k}

(which aren't intersecting), so we can take at

most $k-1$ others - $\mathbb{P} \leq \frac{k}{n}$.

Linearity of Expectation (Chapter 2 in AS)

$x_1, \dots x_n$ are r.v. (real valued random variables) and $c_1, \dots c_n \in \mathbb{R}$ then

$\mathbb{E}(c_1x_1 + \dots + c_nx_n) = c_1\mathbb{E}(x_1) + \dots + c_n\mathbb{E}(x_n)$ - without any ind.

assumption of the x_is.

Excersice - $\mathbb{E}(\text{Fix}(\sigma))$ when $\sigma \in S_n$ is chosen uniformally.

$X = x_1 + \dots + x_n$, $x_i = \mathbb{P}(\sigma(i) = i) = \frac{1}{n}$, so $\mathbb{E}(x) = 1$.

Ex. - largest bipartite subgraph contains at least half of the edges of the original graph.

proof: Put each $v \in V$ in V_1 or V_2 with $\mathbb{P} = 1/2$ each. $\mathbb{E}(|E(V_1, V_2)|) = \frac{|E(G)|}{2}$

This bound can be approved if we avoid very biased partitions.

Prop. Let G be a graph of N vertices and m edges. Then G contains a bipartite subgraph with at least:

(1) $\frac{n}{2n-1} m$ if $N = 2n$, (2) $\frac{n+1}{2n+1} m$ if $N = 2n+1$.

Proof Split V randomly s.t. $|V_1| = |V_2|$, and define a bipartite subgraph H as before. $\mathbb{P}(u \in E(H)) = 2\mathbb{P}(v_1 \in V_1, v_2 \in V_2) = 2\mathbb{P}(v_2 \in V_2 \mid v_1 \in V_1) \cdot \mathbb{P}(v_1 \in V_1) \overset{N=2n}{=}$ $2 \cdot \frac{1}{2} \frac{n}{n+n-1} = \frac{n}{2n+1}$ $\left(\overset{N=2n+1}{=} 2 \cdot \frac{n+1}{2n+1} \cdot \frac{1}{2} = \frac{n+1}{2n+1} \right)$.

Many open questions - what if G is triangle-free, etc.

Ex. Monochromatic k-cliques in 2-colorings of K_n. How many mono. cliques are we guaranteed to see of K_k?

Prop. There is a coloring with at most $\binom{n}{k} 2^{1-\binom{k}{2}}$.

Pf color each edge at random - $\mathbb{E}(\text{mon. cliq.}) = \binom{n}{k} \mathbb{P}(A \text{ is mon. cliq.}) = \binom{n}{k} 2^{1-\binom{k}{2}}$ as required.

Thm. This is asymptotically tight for $k=3$ (Goodman, '59):

$\binom{n}{3} - \lfloor \frac{1}{2} n \lfloor \frac{1}{4}(n-1)^2 \rfloor \rfloor = \frac{1}{4}\binom{n}{3} + \Theta(n^2)$ mon. triangles at least, using a nice double-counting argument.

Conj (Erdős, '62) - This is true for all k - $\left(2^{1-\binom{k}{2}} - o(1)\right)\binom{n}{k}$ at least.

Turns out this isn't true for $k > 3$ (trivial for $k=2$).

Thomason '89 - $\frac{1}{33}\binom{n}{4}$ mon copies of K_4.

Our question for the rest of class - num. of Ham. paths (HP) in a tournament. 1 is possible (and minimal) for the trans. tour. - how large can this number be?

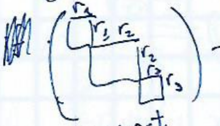
Lower bound - prop. (Szele, '43) there's a tournament with $\geq n! \, 2^{1-n}$ HPs.

pf. A random order is HP with $\mathbb{P} = 2^{1-n}$.

How good is it? Szele's conj. says $\#HP \leq n! (2-o(1))^n$. Proved by Noga Alon - we'll prove it in the rest of the lesson.

If A is a square $n \times n$ matrix define its permanent $\text{per} A = \sum \prod A_{i,\sigma(i)}$

Conj. (Minc)/Thm (Brégman). Suppose $A_{i,j}\in\{0,1\}$. Let $r_i:=\sum_{j=1}^{n} A_{i,j}$. Then

 $\to$ per $A\leq \prod_{i=1}^{n}(r_i!)^{1/r_i}$. Assume per $A>0$, each $r_i>0$.

is worst

Per $A=|S=\{\sigma\in S_n \mid \forall 1\leq i\leq n,\ A_{i,\sigma(i)}=1\}|$. Chose a uniform σ from S and (ind.) $\tau\in S_n$. $L(\sigma,\tau)$ is defined as follows - $A_1=A$, R_1 = #1s in τ_1 of A_1, $A_2=A_1$ after deleting row $\tau(1)$ and col. $\sigma(\tau(1))$. Continue until $A_n=(1)$ in $M_1(\{0,1\})$. $T(\sigma,\tau)$ is $\log R_1\cdot\ldots\cdot R_n$.

claim. per $A\leq \exp(\mathbb{E}[L])$.

pf. We show that for each fixed $\tau\in S_n$, per $A\leq\exp(\mathbb{E}[L|\tau])$. Prove by induction on n. $r=R_1$ = #1s in row $\sigma(1)$. suppose they are in columns $c_1,\dots c_r$. $t_j=\mathrm{Per}(A_{-\sigma(1),c_j})$, $t=\frac{t_1+\dots+t_r}{r}$.

Per $A=t\cdot r$ by definition. $\mathbb{P}(\sigma(\tau(1))=c_j)=\frac{t_j}{tr}$.

$\mathbb{E}(L|\tau)=\log r+\mathbb{E}[L-\log R_1|\tau]$. Condition on c_j to get $\mathbb{E}[L-\log R_1|\tau,\sigma(\tau(1))=c_j]\overset{(ind)}{\geq}\log t_j$. So $\mathbb{E}(L|\tau)=\log r+\sum_{j=1}^{r}\mathbb{P}(\sigma(\tau(1))=c_j)\mathbb{E}[L-\log R_1|\sigma(\tau(1))=c_j,\tau]\geq\log r+\sum_{j=1}^{r}\frac{t_j\log t_j}{tr}\underset{(convexity)}{\geq}$ $\log r+\log t=\log rt=$ per A.

Now, fix $\sigma\in S$ and compute $\mathbb{E}(L|\sigma)$ and take $L=\log R_1'+\dots\log R_n'$ where R_i' = #1s remaining in row i when it's removed. Relabel s.t. $\sigma=id$. r_i ones in i^{th} row, one of them in row i ($\sigma=id$). But many may have been deleted. We removed 1s in a random order - R_i' is uniform on $[r_i]$, so $\mathbb{E}\log R_i'=\frac{1}{r_i}\sum_{j=1}^{r_i}\log j=\frac{\log(r_i!)}{r_i}$.

So, $\exp(\mathbb{E}(L))=\prod_{i=1}^{n}(r_i!)^{\frac{1}{r_i}}\geq$ Per A.

Now we can prove the HP theorem. $P(n)=\max\{\#HP \text{ in } T \mid T \text{ is a tournament}\}$.

Thm (Alon '90) $P(n)=O(n^{3/2}\cdot\frac{n!}{2^{n-1}})$

Pf Given T tournament define its adjacency matrix A_T. So per $A=|$subgraphs of T where each $v\in V$ has exactly 1 in-neighbour and 1 out-neighbour. So $HC(T)\leq \mathrm{Per}\,A_T\leq\prod(r_i!)^{1/r_i}$. So

$C(n) \le \max\limits_{\substack{1 \le r_i \le n-1 \\ \sum_{i=1}^{n} r_i = \binom{n}{2}}} \prod_{i=1}^{n} (r_i!)^{1/r_i}$ T → v' = T' orient at random — $4C(n+1) \geqslant E[\#HC(T')] = \frac{1}{4}\#HP(T)$.

so $4C(n+1) \geqslant P(n)$. The max. is achieved when $|r_i - r_j| \leqslant 1$ for Any i,j. It's enough to show that:

<u>lemma</u> if $b \geqslant a+2$ $(a \geqslant 1)$ then $(b-1)!^{1/b-1}(a+1)!^{1/a+1} > b!^{1/b} \cdot a!^{1/a}$

pf. Let $f(x) = \frac{(x+1)!^{1/x+1}}{x!^{1/x}}$. Enough to show $f(x) \underbrace{\,<\, f(x+1)}_{\text{for } x \geqslant 2}$ to get $f(a) \geqslant f(b)$. Equivalently, we show $x!^{2/x} > (x-1)!^{\frac{2}{x-1}} (x+1)!^{\frac{2}{x+1}} \,/\, x(x+1)(x-1)$

$\Leftrightarrow$

$x!^{2(x^2-1)} > (x-1)!^{2x^2+2x} (x+1)!^{2x^2-2x} \,/\cdot\, (x-1)^{-2x^2}$

$\Updownarrow$

$x!^{-2} x^{2x^2} > x^{x^2-x} (x+1)^{x^2-x} \,/\, x^{-2(x^2-x)}$

$\frac{x^{2x}}{(x!)^2} > \underbrace{\left(1+\frac{1}{x}\right)^{x^2-x}}_{\le e^{x-1}}$. $\left(\frac{x^x}{x!}\right)^2 \geqslant \left(\frac{x^x}{\left(\frac{x+1}{2}\right)^x}\right)^2 = 4^x\left(\frac{x}{x+1}\right)^{2x} = \Theta(4^x)$

Easy to assert for small x.

So $C(n) \leqslant \left(\frac{n-1}{2}!\right)^{\frac{2n}{n-1}} = \Theta\left(\sqrt{n}^{\frac{2n}{n-1}} \left(\frac{n-1}{2e}\right)^{\frac{n-1}{2}\cdot\frac{2n}{n-1}}\right) = \Theta\left(n^2 \cdot 2^{-n}\left(\frac{n}{e}\right)^n\right) =$

$\Theta(\sqrt{n} \cdot 2^{-n} \cdot n!)$. So $P(n) = \Theta(C(n+1)) = \Theta(n^{3/2} \cdot 2^{-n} \cdot n!)$