W3D1 Prob. Methods in Comb.
9/11/14

Recall ~~prop.~~ $\binom{n}{k} 2^{1-\binom{k}{2}} < 1 \to R(k,k) \geq n.$

How to improve that?

<u>Thm</u> $\forall k,n \in \mathbb{N},\ R(k,k) \geq n - \binom{n}{k} 2^{1-\binom{k}{2}}$

<u>Pf.</u> Similar to the original prop, randomly 2-color K_n. For each $R \in \binom{[n]}{k}$ Let $X_R = \mathbb{1}[R \text{ is monochromatic}]$ and $X = \sum_{R \in \binom{[n]}{k}} X_R$.

$\mathbb{E}(X) = \binom{n}{k} 2^{1-\binom{k}{2}}$ and we delete 1 vertex from each clique to get a coloring of size $n - \binom{n}{k} 2^{1-\binom{k}{2}}$.

<u>Remark</u> prop. gives $R(k,k) \geq \left(\frac{1}{\sqrt{2}e} + o(1)\right) k \cdot 2^{k/2}$. The thm gives $\left(\frac{1}{e} + o(1)\right) k 2^{k/2}$.

For $R(k,l)$ we get: $\exists p \in [0,1]$ s.t. $\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}} < 1$ then $R(k,l) > n$.

Thm - $R(k,l) \geq n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l} p^{\binom{l}{2}}$

<u>Example</u> - large ind. sets $\alpha(G) = \max_{A \subseteq V,\, A \text{ ind.}} |A|$

What upper bound on $e(G)$ forces $\alpha(G) \leq t$

$\Updownarrow$

What lower bound on $e(G)$ forces $K_t \subseteq G$?

<u>Thm</u> $e(G) = \frac{nd}{2}$. $\alpha(G) \geq \frac{n}{2d}$ (Turán $\to \alpha(G) \geq \frac{n}{d}$).

<u>pf.</u> fix $0 \leq p \leq 1$, let S be a random subset of $e(G)$, each edge chosen rand. with prob. p. For each S, $\alpha(S) \geq |S| - e(S)$.

$\mathbb{E}|S| = pn$, $\mathbb{E}(e(S)) = \frac{p^2 nd}{2}$. Therefore, $\alpha(G) \geq \max_{0 \leq p \leq 1} pn - \frac{p^2 nd}{2} = n \cdot \max_p \frac{2p - p^2 d}{2}$.

Opt. $p = \frac{1}{d}$ gives $\frac{n}{2d}$.

<u>Thm</u> $\forall G$ - $\alpha(G) \geq \sum_{v \in V} \frac{1}{d_v + 1}$. Sharp for $K_n, \overline{K_n}$.

<u>pf.</u> Let $\leq$ be a rand. ord. of V. Define $I = \{v \in V \mid \forall v' \sim v,\ v \leq v'\}$.

$I(\leq)$ is ind, $\mathbb{E}(I) = \sum_{v \in V} \frac{1}{\deg v + 1}$.

From that we deduce turán's thm. - $\forall n, t \in \mathbb{N}$ let q, r be s.t. $n = tq + r$ $(0 \leq r < t)$. Let $e = r\binom{q+1}{2} + (t-r)\binom{q}{2}$. Define $G_{n,e} = (t-r)K_q + rK_{q+1}$. $\alpha(G_{n,e}) = t$. Then every G' with at most e edges has $\alpha(G') \geq t$. Equality - iff $G' = G_{n,e}$.

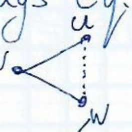
Pf. from prev. thm. First we show that for $G_{n,e}$ the bound is sharp, because every K gives 1 to the sum.

For the 1^{st} part we show that $G_{n,e}$'s deg. seq. gives the smallest sum.

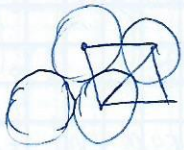
Note that if $v, w \in V$ s.t. $\deg(w) \geq \deg(v)+2$ if we transfer edge from w to v strickly decreases the sum.

For the 2^{nd} part - If G has same deg. seq. but isn't $G_{n,e}$ has larger ind. set.

If $\alpha(G) = t = E(I)$ then I is always eq. to $\sum \frac{1}{\deg v + 1}$. Suppose G isn't a union of cliques -



In that case prove that I isn't const. - For $\underbrace{u \overset{\leq 1}{v} v \ldots}_{\text{order}}$ $\underbrace{v w \overset{\leq 2}{w} \ldots}_{\text{same order}}$. Then $I_{\leq_1} = I_{\leq_2} + 1$.



Graphs with large girth & chromatic num.

$\exists G_n \ \chi(G_n) = n$, $G_n \not\supseteq K_n$.

Perhaps χ is bounded if g is large enough.

$\frac{1}{\sqrt{\frac{\sqrt{3}}{2}}}$ $\qquad \frac{\frac{1}{\sqrt{\pi/2}}}{\sqrt{3}} = \frac{2\pi}{2^{\sqrt{3}}} 10^{2} > 9$

Thm. $\forall k, l \ \exists G$ s.t. $\chi(G) > k$, $girth(G) > l$.

pf Fix $0 < \varepsilon < \frac{1}{l}$. Let $G \sim G(n,p)$, $p = n^{\varepsilon - 1}$. Let $X = \#$cycles of length $\leq l$.

$E(X) = \sum_{i=3}^{l} \frac{(n)_i}{2i} p^i \leq \sum_{i=3}^{l} n^i p^i = O(n^{l\varepsilon}) = o(n)$. We use $\chi \leq \frac{|V|}{\alpha}$.

$P(\alpha(G) \geq x) = P\left(\bigcup_{|A|=x} \{e_G(A) = 0\}\right) \leq \sum_{|A|=x} P(e_G(A) = 0) = \binom{n}{x}(1-p)^{\binom{x}{2}} \leq n^x e^{-p\binom{x}{2}} =$
$\left(n e^{-\frac{p(x-1)}{2}}\right)^x$. So, $x \sim \frac{\log n}{p}$. Let $x = \lceil \frac{4}{p} \log n + 1 \rceil$. This is $\Theta(n^{1-\varepsilon} \log n)$. So $P(\alpha(G) \geq x) \leq n^{-x} = o(1)$.

If $P(X > \frac{n}{2})$ is small we'll get $P(\alpha(G) < 10 n^{1-\varepsilon} \log n \cap X \leq \frac{n}{2}) > 0$
$\qquad = 1 - o(1)$. By markov, $P(X > \frac{n}{2}) \leq \frac{E[X]}{n/2} = o(1)$. and we delete one vertex from each of the X cycles in G.

Two colorability of uniform hypergraph.

$m(n) = \{ \min(e(\mathcal{H})) \ \mathcal{H}$ is a non-2-col. n-uniform hypergraph$\}$

we showed $m(n) \geq 2^{n-1}$, and later $m(n) \leq \left[\frac{e \log 2}{4} + o(1)\right] n^2 2^n$.

Turns out $m(n) \geq \Omega(n^{1/3} 2^n)$ and even $m(n) = \Omega\left(\sqrt{\frac{n}{\log n}}\, 2^n\right)$:

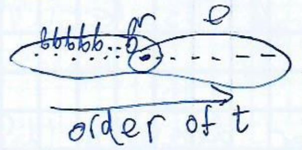
$\forall c < \sqrt{2}$, $m(n) \geq c\sqrt{\frac{n}{\log n}}\, 2^{n-1}$ for $n > n_0(c)$. Take $\mathcal{H}$ the n-uniform hypergraph with $k 2^{n-1}$ edges. Always color blue (by a random order) unless you can't, in which case color red. This is equiv. to choosing $t(v) \in [0,1]$ uniformally and adding them by the t-order.

WLG assume t is 1-1. suppose $v_1 \leq \dots \leq v_n$. Do the following -

For $1 \leq i \leq n$ - color v_i blue if you can, o/w color red.

Proper $\Leftrightarrow$ no red edge. Prove (under assumptions) $\mathbb{P}(\text{red edge} < 1)$.

If $e \in E$ becomes red, so



Call a pair e, f <u>conflicting</u> if

$|e \cap f| = 1$ and $e \leq f$. $\mathbb{P}(\text{red edge}) \leq \mathbb{P}(\text{conf. pair})$.

Split $[0,1]$ to 3 intervals $L = [0, \frac{1-p}{2})$, $M = [\frac{1-p}{2}, \frac{1+p}{2}]$, $R = (\frac{1+p}{2}, 1]$.

We look at 3 events for each of $i = L, M, R$, - A_i - $\exists$ conf pair with $x \in e \cap f$ in i. By symm., $\mathbb{P}(A_L) = \mathbb{P}(A_R) \leq \sum_{e \in \mathcal{H}} \mathbb{P}(t(e) \subseteq L) = k 2^{n-1} \left(\frac{1-p}{2}\right)^n$. Hence $\mathbb{P}(A_L \cup A_R) \leq k(1-p)^n$. OTOH,

$$\mathbb{P}(A_M) \leq \mathbb{E}[\#\text{ conf. pairs in } A_M] \leq k^2 2^{2n-2} \int_{\frac{1-p}{2}}^{\frac{1+p}{2}} x^{n-1} (1-x)^{n-1}\, dx = k^2\, 2 \int_{-\frac{p}{2}}^{\frac{p}{2}} \underbrace{(1-2y)^{n-1}(1+2y)^{n-1}}_{\leq 1} dy \leq k^2 \cdot p.$$

(pairs $e \leq f$)

So $\mathbb{P}(\text{red edge}) \overset{?}{\leq} k(1-p)^n + k^2 p \leq k e^{-pn} + k^2 p$. Set $k = c\sqrt{\frac{n}{\log n}}$, $p = \frac{\log \frac{n}{k}}{n} = o(1)$.

$k e^{-pn} = \frac{k^2}{n} = O(\log n^{-1}) = o(1)$. $k^2 p = c^2 \dfrac{\log \left(\frac{\sqrt{n \log n}}{c}\right)}{\log n} \xrightarrow[n \to \infty]{} \frac{c^2}{2} < 1$.

Hence $\mathbb{P}(\text{red edge}) < 1$ as required

<u>Example from last time (Lin. of Exp.)</u>

<u>Thm</u> Let $v_1, \dots, v_n \in \mathbb{R}^d$ with norm 1. $\exists\, \varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$ s.t. $\|\sum_{i=1}^n \varepsilon_i v_i\| \leq \sqrt{n}$. Sharp for $e_1 \dots e_d$.

<u>Pf</u> choose ε_i uni. ind., $\mathbb{E}(\|\sum_{i=1}^n \varepsilon_i v_i\|^2) = \mathbb{E}[\langle \sum \varepsilon_i v_i, \sum \varepsilon_i v_i \rangle] = \sum \langle v_i, v_i \rangle = n$.