WSD1
23/11/14 Prob. Methods in Comb.

Local Lemma

We have $A_1, \ldots, A_n$ ~~events~~ we want to avoid. If they're indep. $\mathbb{P}(\overline{A}_1 \cap \overline{A}_2 \ldots \cap \overline{A}_n) = \prod_{i=1}^{n} \mathbb{P}(\overline{A}_i)$. The local lemm provides weaker conditions for which we can still claim something similar.

Thm. (Symm. Local Lemma) Let $A_1, \ldots A_n$ be event in an prob. space. Suppose each A_i is mutually indep. on all but of ~~most~~ d events. If $\mathbb{P}(A_i) \leq p$ for any i and $e(d+1)p \leq 1$ then $\mathbb{P}(\bigcap_{i=1}^{n} \overline{A}_i) > 0$

Before we prove, some examples.

Example (1): 2-col. of uniform hypergraphs

Prop. Suppose each edge of an n-uniform hypergraph H intersects at most d other edges. If $e(d+1) \leq 2^{n-1}$, then H is 2-col.

Pf. 2-color $V(H)$ at random ($p = \frac{1}{2}$ for each color), and apply the local lemma.

Example (2): Ramsey Numbers

Prop. If $e\binom{k}{2}\binom{n-2}{k-2}2^{1-\binom{k}{2}} < 1$ then $R(k,k) > n$.

Pf Uniformly 2-color the edges of K_n. Apply the local lemma. The trivial bound was for $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ $[\rightarrow R(k,k) > n]$. This is roughly $\left(\frac{en}{k}\right)^k 2^{1-\binom{k}{2}} = \left(\frac{en 2^{-\frac{k-1}{2}}}{k}\right)^k \overset{?}{<} 1$, so $n \leq \frac{k 2^{k/2}}{e\sqrt{2}}(1+o(1))$. The bound given by the local lemma is roughly $\left(\frac{ne}{k}\right)^{k-2} k^2 2^{\frac{-k(k-1)}{2}} < 1$, or equiv. $\frac{ne}{k} 2^{\frac{-k(k-1)}{2(k-2)}} < 1$, so $n \leq \frac{k}{e} 2^{\frac{k}{2}+\frac{1}{2}}$. So, we get an improvement by a factor of 2.

Consider $R(k,3)$. 2-Color K_n randomly. $\mathbb{P}(ij \text{ red})=1-p$, $\mathbb{P}(\text{blue})=p$.
We try and avoid blue K_3 / red K_k. For a given T,
$\mathbb{P}(\underbrace{T \text{ is } B}_{A_T})=p^3$ and similarly $\mathbb{P}(\overbrace{S \text{ is red}}^{A_S})=(1-p)^{\binom{k}{2}}$. We need
a more general version of the LL.

Let $A_1, \dots A_n$ as usual. Define a dependency graph $\underbrace{A_i \text{ indep. of } \{A_j\}_{e_{ij} \notin E}}$
on $\{1, \dots, n\}$, for $A_1, \dots, A_n$. This isn't canonical:
Let $\{x_1, \dots, x_n\} \in \{0,1\}^n$ be chosen from the set of all seq.
$x_1 + \dots + x_n \equiv 0\ (2)$. Let $A_i = "x_i = 0"$. Then there's no min.
dependency graph.

<u>Thm (Gen. LL)</u>

Let $A_1, \dots, A_n$ with dep. digraph G. If $\exists x_1, \dots x_n \in (0,1)$
s.t. $\mathbb{P}(A_i) \le x_i \prod_{\vec{e_{ij}} \in E}(1-x_j)$, then $\mathbb{P}(\bigcap_{i=1}^{n} \bar{A}_i) \ge \prod_{i=1}^{n}(1-x_i) > 0$.
If $\forall i\ |\{j : \vec{e_{ij}} \in E\}| \le d$ we let
$x_1 = \dots = x_n = \frac{1}{d+1} < 1$. (we clearly may assume $d>0$).
$\mathbb{P}(A_i) \underset{\text{ass.}}{\le} \frac{1}{(d+1)e} \le x_i \prod_{\vec{e_{ij}} \in E}(1-x_j)$ so we get the symm. version.

(since $(1-\frac{1}{d+1})^d \ge \frac{1}{e}$)

We can now return to bounding $R(k,3)$. A dependency graph on $V(G) = \binom{[n]}{3} \cup \binom{[n]}{k}$ with $X \sim_G Y$ iff $|X \cap Y| \ge 2$.

<u>Check</u>: $p^3 \le x \cdot (1-x)^{3(n-3)} (1-y)^{\binom{n}{k}}$, with $x_T = x$, $x_S = y$. Also, check
$(1-p)^{\binom{k}{2}} \le y \cdot (1-x)^{\binom{k}{2}n} (1-y)^{\binom{n}{k}}$. $\exists$ such x, y in $(0,1)$ $\to R(3,k) > n$.

Turns out one may take $p = c_1 n^{-1/2}$, $\underbrace{k = c_2 n^{1/2} \log n}_{\text{gives the lower bound}}$,
$x = c_3 n^{-3/2}$, $y = c_4 \binom{n}{k}^{-1}$

This implies $R(3,k) \ge c\frac{k^2}{\log^2 k}$. How good is this? Turns out $R(3,k) \le \frac{Ck^2}{\log k}$, which was proved to be tight.

One can in fact, take $c = \frac{1}{4} + o(1)$, $C = 1$.

Now let's prove the lemma.

pf. $\mathbb{P}(\bigcap_{i=1}^{n} \bar{A}_i) = \mathbb{P}(\bar{A}_1)\mathbb{P}(\bar{A}_2 | \bar{A}_1) \dots \mathbb{P}(A_n | A_1, \dots, A_{n-1}) = (1-\mathbb{P}(A_1)) \dots (1-\mathbb{P}(A_n | \bar{A}_{n-1} \dots \bar{A}_1))$

Its enough to show $\forall i\ \mathbb{P}(A_i \mid \overline{A}_1, \overline{A}_2, \dots \overline{A}_j) \leq x_i$.

We prove by induction the stronger statement $\forall S \subseteq [n]\ \ p(S): \forall i \notin S,\ \mathbb{P}(A_i \mid \bigcap_{j\in S} \overline{A}_j) \leq x_i$, on the size of S. $S=\emptyset$ is trivial. Assume $S \neq \emptyset$ and take $S_1 = \{j \in S \mid \overrightarrow{e_{ij}} \in E\}$ and $S_2 = S \setminus S_1$. $\mathbb{P}(A_i \mid \bigcap_{j\in S} \overline{A}_j) = \dfrac{\mathbb{P}(A_i \cap \bigcap_{j\in S_1} \overline{A}_j \mid \bigcap_{j\in S_2} \overline{A}_j)}{\mathbb{P}(\bigcap_{j\in S_1} \overline{A}_j \mid \bigcap_{j \in S_2} \overline{A}_j)}$. Now, $\mathbb{P}(A_i \cap \bigcap_{j\in S_1} \overline{A}_j \mid \bigcap_{j\in S_2} \overline{A}_j) \leq \mathbb{P}(A_i \mid \bigcap_{j\in S_2} \overline{A}_j) \underset{A \text{ ind. of } \{A_j\}_{j\in S_2}}{=} \mathbb{P}(A_i) \leq x_i \prod_{e_{ij}\in E} (1-x_j)$

and $\mathbb{P}(\bigcap_{j\in S_1} \overline{A}_j \mid \bigcap_{j\in S_2} \overline{A}_j) \overset{S_1 = \{i_1, \dots i_r\}}{=} [1 - \mathbb{P}(A_{i_1} \mid \bigcap_{j\in S_2} \overline{A}_j)] \dots [1 - \mathbb{P}(A_{i_r} \mid \overline{A}_{i_1} \cap \dots \overline{A}_{i_{r-1}} \bigcap_{j\in S_2} \overline{A}_j)]$

and we may use the inductive assumption to get that this $\geq (1-x_{i_1}) \dots (1-x_{i_r}) = \prod_{i=1}^{r} (1-x_{i_r}) \geq \prod_{e_{ij}\in E} (1-x_j)$, so we get $\mathbb{P}(A_i \mid \bigcap_{j\in S} \overline{A}_j) \leq x_i$.

"lopsided" local lemma

So, we see it's enough that $\mathbb{P}(A_i \mid \bigcap_{j\in S_2} \overline{A}_j) \leq \mathbb{P}(A_i)$ to prove the same result (A_i is negatively correlated with $\bigcap_{j\in S_2} \overline{A}_j$ for any $S_2 \subseteq \{j \mid \overrightarrow{e_{ij}} \notin E\}$.

Consider $c: \mathbb{R} \to \{1, \dots, k\}$. We say $\underbrace{T \subseteq \mathbb{R}}_{\text{is multicolored}}$ if $c(T) = \{1, \dots, k\}$.

Thm. Suppose $e((m-1)m+1)k(1-\frac{1}{k})^m < 1$ (roughly, $m > (3+o(1))k \log k$)

Then $\forall S \subseteq \mathbb{R}$ with at least m elements $\exists c$ a coloring of $\mathbb{R}$ such that $\forall x$, $S+x$ is multicolored.

pf. First fix some $X \subseteq \mathbb{R}$ finite and prove for $\forall x \in X$. For that, color i.i.d. uniformly. Define $A_x =$ "$x+S$ isn't multicolored". Then $\mathbb{P}(A_x) \leq k(1-\frac{1}{k})^m$

A_x is ind. of $\{A_{x'}\}_{(S+x')\cap(S+x)=\emptyset}$ and there are at most $m(m-1)$ x' that won't satisfy this.

So, we get the result from LL.

How can we deduce a coloring for $X = \mathbb{R}$?

consider $\Omega = \{\text{all colorings}\} = \{1,\dots,k\}^{\mathbb{R}}$ with a topology – the Tychonoff topology (product topology).

<u>Thm.</u> (Tychonoff) Ω is compact.

So, for each $x \in \mathbb{R}$ define $T_x = \{\text{colorings with } S+x \text{ multicolored}\}$ closed, as a finite union of cylinders. We know that for any finite $X \in \mathbb{R}$, $\bigcap_{x \in X} T_x \neq 0$, so (from compactness) $\bigcap_{x\in\mathbb{R}} T_x \neq \emptyset$.

$A = (a_{ij})$ is a $n \times n$ matrix. $\pi \in S_n$ is a <u>transversal</u> if $a_{i,\pi(i)}$ are distinct.

<u>Thm.</u> Suppose no entry of A appears more than $\frac{n-1}{4e}$ $(=k)$ times. Then A has a transversal.

<u>pf</u> Let π be a uniform element of S_n. Define "bad events", $T = \{(i,j,i',j') \mid a_{ij} = a_{i'j'},\ i<i',\ j \neq j'\}$ $B_t = $ "$\pi_i = j$, $\pi_{i'} = j'$", $t \in T$.

$\mathbb{P}(B_t) = \frac{1}{n(n-1)}$. Define G with $V(G) = T$ with $(i,j,i',j') \sim_G (p,q,p',q') \iff \{i,i'\} \cap \{p,p'\} \neq \emptyset$ or $\{j,j'\} \cap \{q,q'\} \neq \emptyset$. So $\Delta(G) \leq 4n \cdot k$. By ass. $\frac{1}{n(n-1)} 4nk \cdot e \leq 1$.

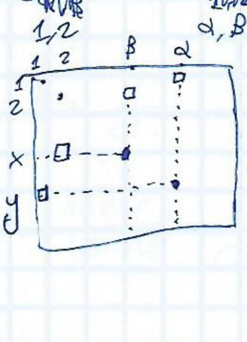
We can conclude <u>if</u> we show "lopsided" property.

WLG assume $i=j=1$, $i'=j'=2$. Let $S \subseteq T$ s.t. $\forall (p,q,p',q') \in S$ with $p,q,p',q' \geq 3$. $\mathbb{P}(B_{(1,1,2,2)} \mid \bigcap_{t\in S} \overline{B_t}) \leq \frac{1}{n(n-1)} = \mathbb{P}(B_{(1,1,2,2)})$.

Let $S_{\alpha,\beta} = \{\pi \in S_n \mid \pi(1) = \alpha,\ \pi(2) = \beta,\ \pi \in \bigcap_{t\in S} \overline{B_t}\}$. $\bigcup_{\alpha,\beta} S_{\alpha,\beta} = \bigcap_{t \in S} \overline{B_t}$.

Enough to show $|S_{\alpha,\beta}| \geq |S_{1,2}|$ for each α, β, since $\mathbb{P}(B_{(1,1,2,2)} \mid \bigcap_{t\in S} \overline{B_t}) = \frac{|S_{1,2}|}{\sum |S_{\alpha,\beta}|}$. We define an injection $S_{1,2} \hookrightarrow S_{\alpha,\beta}$ to show that. Assume $\{\alpha,\beta\} \cap \{1,2\} = \emptyset$.

other cases are simpler...



• – original $\pi \in S_{1,2}$

▫ – new $\pi \in S_{\alpha,\beta}$.

Corollary On any infinite network $\exists$ coupling (F_1, F_2) of WSUF and FSUF s.t. $F_1 \subseteq F_2$ a.s.

Cor. 2 If $\mathbb{E}$ deg x is the same in FSF, WSF, then $\mu^F = \mu^W$.

pf. If the condition holds in the monotone coupling the edges incident to x in WUSF & FUSF are equal with prob. 1, so $\mu^F = \mu^W$.

Defs. G graph $K \subset V$ finite subset, the edge boundary $\partial_E K = \{e \in E \mid \underset{(x,y)}{} \; {x \in K \atop y \notin K}\}$. G is edge-ameanable iff $\exists$ finite $V_n \subset V$ with $\frac{|\partial_E V_n|}{|V_n|} \to 0$.

Example - $\mathbb{Z}^d$ with V_n ball of radius n, since $|V_n| = \Theta(n^d)$ and $|\partial_E V_n| = \Theta(n^{d-1})$.

Non-example: 3-reg. tree.

We'll see edge-ameanable $\to \mu^F = \mu^W$.