Random matrices - lecture #1 24/10/2011

Theme:

$$A = \underbrace{\begin{pmatrix} \text{matrix} \end{pmatrix}}_{N} \Big\} N \quad , A \text{ is random}$$

study the eigenvalues & eigenvectors of $A$.

Type of theorems we'll see:

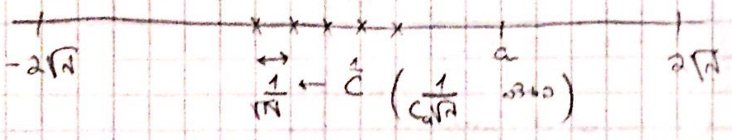
Random matrix $A$ will be Hermitian, $A = A^*$

$\Longrightarrow$ eigenvalues are real.

Wigners model:

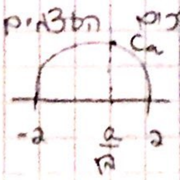
$A =$  the entries are Independent, identically dist. except for being Herm (meaning the upper triangle is distributed as mentioned)

particularly important: Gaussian entries. $N(0,1)$




כה נכון עד כדי הקבוע המוינה ויוצא שהמרחקים הטבעיים יותר צפופים במרכז.

1) וכעת הם מתפלגים לפי חצי מעגל:

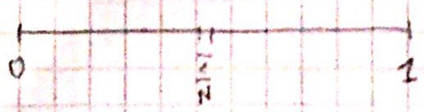



this is Wigners semi-circle distributed.

המספר הממוצע בקטע $(c,d)$ הוא כמו השטח מתחת לחצי המעגל של Wigner


another Problem:

N uniform independent point




$$P\left(\begin{matrix}\text{to see 2 points in an}\\ \text{interval of length } \frac{\varepsilon}{N}\end{matrix}\right) \approx \varepsilon^2$$

2) however, Eigenvalues have repulsion, same Prob. will be $\varepsilon^{\alpha}$, $\alpha > 2$

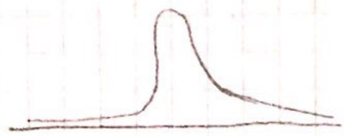
these will lead us to: $\Longrightarrow$ Determinantal Process, Sine Kernel

3) Maximal eigenvalues.

$\lambda$ - max eigenvalue

$E(\lambda) \sim 2\sqrt{N}$

$std(\lambda) \sim cN^{1/6}$



$\frac{\lambda - E(\lambda)}{std(\lambda)} \xrightarrow{d}$ Tracy - Widom distribution

Gaussian distribution:

a mathematical application:

Longest increasing subsequence.

Permutation $\pi \in S_N$ chosen uniformly.

for example: $N = 7$ : $3\,5\,1\,6\,2\,7\,4$ $\quad L(\pi) = 4$

-, ~ : these are increasing subsequences

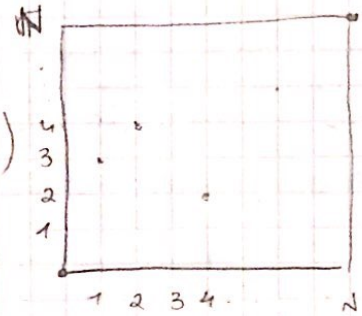
$L(\pi)$ - length of the longest increasing subsequence

Ulam 1960:

$E(L(\pi)) \sim 2\sqrt{N}$ (Vershik - kerov)

$std(L(\pi)) \sim cN^{1/6}$ (Beik - Deift - Johanson)

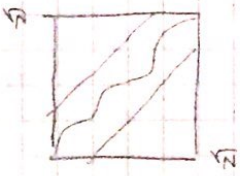
$\frac{L(\pi) - E\,L(\pi)}{std(L(\pi))} \xrightarrow{d}$ Tracy widom




the longest increasing subsequence is equivalent to finding the longest path from $(0,0)$ to $(n,n)$ (as in, the path going through as many points as possible).

אפשר להניח שבמקום בוחרים N נקודות אקראיות, שהתפלגות יהיה קרוב לפואסון

N uniform points.




שהוא מאורך $\sqrt{N}$ (סדר גודל)

Random operators:

$$\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & 0 \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ 0 & & & & \ddots \end{pmatrix} \text{ - Laplacian}$$

this matrix corespunds to the probability of moving to one of the adjacent cells.

Anderson Localization:

$$\begin{pmatrix} V_1 & 1 & & & \\ 1 & V_2 & 1 & & \\ & 1 & V_3 & 1 & \\ & & & \ddots & 1 \\ & & & 1 & V_n \end{pmatrix} \quad V_i \text{ are IId } N(0,1)$$

the question is how do the random variables effekt the movement of the particals? turns out that the Eigenvectors where the particle is" in physics you want to study the position of the particle - will it be localized (stuck) or will its position change - delocalized.

Herm. there are $W$ diaginals of IId, $N(0,1)$ random variable. this is the Band Matrix.
when $W \ll \sqrt{N}$ the position of the partical is localized.

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Book: Anderson, Guionnet, Zeitouni, an introduction to random matrices.

Grade: based on homeworks.

## Wigner model:

Two families of IID $\overset{\text{real}}{\vee}$ random Var. $(Z_{i,j})_{i>j\geq 1}$, $(Y_i)_{i\geq 1}$

IID ↗    ↖ independent ↗    ↖ IID

Random matrix $X_N$, $N\times N$ matrix.

$$X_N = \begin{array}{c} \\ i \\ j \end{array}\begin{array}{c} \qquad i \qquad\quad j \\ \begin{pmatrix} Y_1 & & \vdots & & \\ \cdots & \ddots & & Z_{i,j} & \\ \cdots & Z_{i,j} & & \ddots & \\ & & & & Y_N \end{pmatrix} \end{array} \cdot \frac{1}{\sqrt{N}} \quad \text{Symmetric}$$

↖ here we scale the eigenvalues of this matrix.

### Assumptions:

1) $\max\left(\mathbb{E}(|Z_{i,j}|^k), \mathbb{E}(|Y_i|^k)\right) < \infty$ : $k$ לכל

2) $\mathbb{E}(Z_{i,j}) = 0$, $Var(Z_{i,j}) = 1$

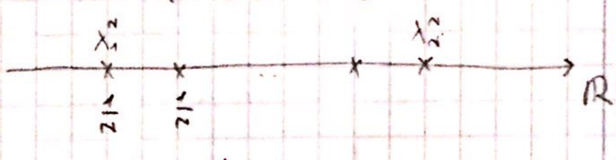
3) $\mathbb{E}(Y_i) = 0.$

we study the empirical eigenvalue dist:

$\lambda_1^N \leq \dots \leq \lambda_N^N$ eigenvalues.

$L_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i^N}$

כוה מידה מהדרה הבטוה:




$f:\mathbb{R}\to\mathbb{R}$ function:

$\langle L_N, f\rangle = \int f(x)\,dL_N(x) =$

$= \frac{1}{N}\sum_{i=1}^{N} f(\lambda_i^N)$

**Semi-circle:** measure on $\mathbb{R}$ given by: $d\sigma = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbb{1}_{[-2,2]}\,dx$

(הגדרנו מידה עם פונק"ת צפיפות סימטרית, סוחס מאודרת סבס)

$\langle \sigma, f\rangle = \int f(x)\,d\sigma(x) = \int_{-2}^{2} f(x)\cdot\frac{1}{2\pi}\sqrt{4-x^2}\,dx$

we will want to prove that the $L_N$ measure converges to $\sigma$

### Reminder:

A $\overset{\text{sequence of}}{\vee}$ probability measures $(\mu_n)$ on $\mathbb{R}$ converges to a probability measure $\nu$ on $\mathbb{R}$ (weak convergence, in distribution) if for every bounded continuous function $f$ on $\mathbb{R}$:

$$\int f(x)\,d\mu_n(x) \xrightarrow[n\to\infty]{} \int f(x)\,d\nu(x)$$

there is a slight twist here for $\mathcal{L}_N$ is a Random measure.

Theorem (Wigner):

$\mathcal{L}_N$ converges to $\sigma$ weakly in probability in the sense that:

$\forall f$ bounded, continuous & $\forall \varepsilon > 0$,

$$P\left( \left| \int f(x)\, d\mathcal{L}_N(x) - \int f(x)\, d\sigma(x) \right| > \varepsilon \right) \xrightarrow[N \to \infty]{} 0$$

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Basic tool:
$$\mathrm{tr}(X_N) = \sum_{i=1}^{N} \lambda_i^N$$
$$\mathrm{tr}(X_N^k) = \sum_{i=1}^{N} (\lambda_i^N)^k$$

Method of moments:

$(\mu_N)$ prob. measures, $\nu$ prob. measures.

given: $\int x^k\, d\mu_N(x) \xrightarrow[N \to \infty]{} \int x^k\, d\nu(x) \quad \forall k \geq 1$ int.

Hausdorff: $\nu$ of bounded support Implies $\mu_N \xrightarrow{d} \nu$

general $\nu$: Not necessarily, but yes if $\int x^k\, d\nu$ grows "not too fast" with $k$.

Moments of semi-circle law

$$m_k := \langle \sigma, x^k \rangle = \int x^k\, d\sigma(x) \quad \forall k \geq 0 \text{ int.}$$

Proposition:

$m_k = 0$ if $k$ is odd. $m_{2k} = \underset{\text{catalan number}}{C_k} = \frac{1}{k+1}\binom{2k}{k} = \frac{2k!}{k!(k+1)!}$

Proof:

$m_k = 0$ for $k$ odd, by symmetry of $\sigma$.

for $k \geq 1$, establish a recursion:

$$m_{2k} = \int_{-2}^{2} x^{2k} \frac{1}{2\pi} \sqrt{4 - x^2}\, dx \underset{\substack{x = 2\sin\theta \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^2 \frac{2^{2k}}{2\pi} \sin^{2k}(\theta) \cos^2(\theta)\, d\theta \underset{\cos^2\theta = 1 - \sin^2\theta}{=}$$

$$= \frac{2}{\pi} \cdot 2^{2k} \left[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k}(\theta) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k+2}(\theta) \right] = (*)$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k+2}(\theta)\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overset{\text{diff}}{\sin^{2k+1}(\theta)}\, \overset{\text{int}}{\sin(\theta)}\,d\theta =$$

$$= +(2k+1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k}(\theta)\cos^2(\theta)\,d\theta = \frac{\pi}{2\cdot 2^{2k}}(2k+1)\,m_{2k}$$

$$\Longrightarrow (*) = \frac{2}{\pi}\,2^{2k}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k}(\theta) - (2k+1)\,m_{2k}$$

$$\Longrightarrow m_{2k} = \frac{2}{\pi(2k+2)}\,2^{2k}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2k}(\theta) \overset{\text{as before}}{=} \frac{4(2k-1)}{2k+2}\,m_{2k-2}$$

have also $m_0 = 1$. Check that Catalan numbers satisfy this.