Lecture 8:

12/12/2017

Reminder:

GUE random matrix

$$N\left\{\overbrace{\begin{pmatrix} \xi_{11} & \frac{\xi_{12}+i\eta_{12}}{\sqrt{2}} & \\ & \ddots & \\ \text{Herm} & & \xi_{NN} \end{pmatrix}}^{N}\right.$$

ξ, η IID $N(0,1)$, $\lambda_1^N \le \dots \le \lambda_N^N$

$\mu_1, \dots, \mu_N$ for the unordered eigenvalues - applying a uniform permutation to $(\lambda_1^N, \dots, \lambda_N^N)$.

Lemma:

joint density of $\mu_1, \dots, \mu_p$ is:

density: $\frac{(N-p)!}{N!} \det_{i,j=1}^{p} K^{(N)}(x_i, x_j)$

Kernel: $K^{(N)}(x,y) = \sum_{k=0}^{N-1} \psi_k(x)\psi_k(y)$

Hermite Poly:

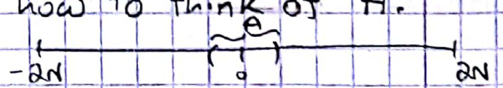
$$h_n(x) := (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$$

Oscilator wave functions:

$$\psi_n(x) := \frac{e^{-\frac{x^2}{4}} h_n(x)}{(\sqrt{2\pi}\, n!)^{1/2}}$$

Lemma: (gap prob.)

how to think of A:



For any meas. $A \subseteq \mathbb{R}$:

$$P\left(\bigcap_{i=1}^{N}(\sqrt{N}\lambda_i \notin A)\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{A/\sqrt{N}} \cdots \int_{A/\sqrt{N}} \det_{i,j=1}^{k}\left(K^{(N)}(x_i, x_j)\right) \prod_{i=1}^{k} dx_i$$

Fredholm determinant.

Thm. (Gaudin - Mehta):

For A compact:

$$\lim_{N\to\infty} P\left(\bigcap_{i=1}^{N}(\sqrt{N}\lambda_i \notin A)\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_A \cdots \int_A \det_{i,j=1}^{k}\left(S(x_i, x_j)\right) \prod_{i=1}^{k} dx_i$$

Sine kernel: $S(x,y) := \frac{1}{\pi} \frac{\sin(x-y)}{x-y}$

Main step in proof:

Define $S^{(N)}(x,y) := \frac{1}{\sqrt{N}} K^{(N)}\left(\frac{x}{\sqrt{N}}, \frac{y}{\sqrt{N}}\right)$

Lemma:

$\lim_{N\to\infty} S^{(N)}(x,y) = S(x,y)$ uniformly for x,y in any given compact set.

Missing still:

- Prove lemma in main step.
- Prove continuity of fredholm determinants.

Fredholm determinants:

Let X be a locally compact polish space (e.g. $X = \mathbb{R}^d$) & ν a complex measure on X (with Borel σ-algebra), e.g. Lebesgue measure on an interval in $\mathbb{R}$.

Assume: $\|\nu\|_1 := \int_X |\nu| dx < \infty$

Def:

A kernel is a Borel-measurable function:

$K: X\times X \to \mathbb{C}$ s.t. $\|K\| := \sup_{x,y\in X} |K(x,y)| < \infty$

Aside: Hadamard inequality:

For any $k\times k$ matrix A, $|\det(A)| \le \prod_{i=1}^{k} \|A_i\|_2$ ← rows of A

$\Rightarrow |\det A| \le \max_{i,j} |A_{ij}|^k \cdot k^{k/2}$

Proof:

$\det A = \prod_{i=1}^{k} \|A_i\|_2 \det B$ where $B_i := A_i / \|A_i\|_2$

Let $m := B\cdot B^*$ so that diagonal entries of m are 1 & m is positive semi-definite.

$\det(m) = \prod_{i=1}^{k} \underbrace{\lambda_i(m)}_{\text{eigenvalues}} \le \left(\frac{1}{k}\sum_{i=1}^{k} \lambda_i(m)\right)^k = \left(\frac{1}{k}(\mathrm{tr}(m))\right)^k = 1$

arithmetic-geometric mean.

$$\Rightarrow \det(A) = \prod_{i=1}^{k} \|A_i\|_2 \cdot \underbrace{\sqrt{\det M}}_{\leq 1}$$

<u>Lemma:</u>

For any two kernels K, L : $\forall x_1, \dots, x_k \in X$

$$\left| \det_{i,j=1}^{k} K(x_i, x_j) - \det_{i,j}^{k} L(x_i, x_j) \right| \leq k^{1+k/2} \max(\|K\|, \|L\|)^{k-1} \|K - L\|$$

As a special case of the proof:

$$\left| \det_{i,j=1}^{k} K(x_i, x_j) \right| \underset{\uparrow}{\leq} k^{k/2} \|K\|^{k}$$

Hadamard inequality

<u>Proof:</u>

$$\det \begin{pmatrix} K(x_1,x_1) & K(x_1,x_2) & \cdots \\ \vdots & & \\ \vdots & & \end{pmatrix} - \det \begin{pmatrix} L(x_1,x_1) & L(x_1,x_2) & \cdots \\ \vdots & & \\ \vdots & & \end{pmatrix} =$$

$$= \sum_{i=1}^{k} \det \begin{pmatrix} L(x_1,x_1) & L(x_1,x_2) & \cdots \\ \vdots & & \\ L(x_{i-1},x_1) & \cdots & \\ (K-L)(x_i,x_1) & \cdots & \\ K(x_{i+1},x_1) & \cdots & \\ \vdots & & \end{pmatrix} \leq k \max(\|K\|, \|L\|)^{k-1} \|K-L\| k^{k/2}$$

<u>Def:</u>

The Fredholm determinant of K, ν:

$$\Delta(K) = \Delta(K, \nu) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Delta_k(K, \nu)$$

where $\Delta_k(K, \nu) = \int_X \dots \int_X \det_{i,j=1}^{k} K(x_i, x_j) \prod_{i=1}^{k} d\nu(x_i)$

where $\Delta_0(K, \nu) := 1$

Prev. lemma shows it is well defined (absolute convergence of sum)

<u>Conclusion:</u>

For two kernels K, L:

$$|\Delta(K) - \Delta(L)| \leq \sum_{k=1}^{\infty} \frac{k^{1+k/2} \|\nu\|^k \max(\|K\|, \|L\|)^{k-1}}{k!} \|K - L\|$$

$$\Delta(K) - \Delta(L) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\Delta_k(K) - \Delta_k(L) \right)$$

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Remark: If K has rank n, $K(x,y) = \sum_{i=1}^{n} f_i(x) g_i(x)$

$$\Delta(K) = \det_{i,j=1}^{n} \left(I_{k \times k} - \int f_i(x) g_j(x) \, d\nu(x) \right)$$

$$\text{"} \Delta(K) = \det(I - K) \text{"}.$$

Additional Prop. of Hermite Poly.:

Reminder: $h_0(x) = 1$, $h_{n+1}(x) = x h_n(x) - h_n'(x)$

$$\int h_k(x) h_\ell(x) e^{-x^2/2} \frac{1}{\sqrt{2\pi}} dx = k! \, \delta_{k,\ell}$$

Lemma:

1) $x h_n(x) = h_{n+1}(x) + n \cdot h_{n-1}(x) \quad \forall n \geq 1$. (Three-term recurrence)

2) $h_n'(x) = n h_{n-1}(x)$

3) $\displaystyle\sum_{k=0}^{n-1} \frac{h_k(x) h_k(y)}{k!} = \frac{h_n(x) h_{n-1}(y) - h_{n-1}(x) h_n(y)}{(n-1)! \quad x-y}$ for all $x \neq y$

(Christoffel - Darboux formula)

Proof:

(2) follows from (1) & the recursion.

For (1), develop $x h_n(x)$ in the basis $h_0, \dots, h_{n+1}$

$$x h_n(x) = \sum_{k=0}^{n+1} \frac{1}{k!} \int y h_n(y) h_k(y) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \; h_k(x)$$

As span $(h_0, \dots, h_m)$ = Poly. of deg m, it follows that k'th term is 0 when $n > k+1$.

Remaining terms have $k = n-1$, $k = n$, $k = n+1$. When $k = n$, $y h_n(y)$ & $h_n(y)$ are even & odd functions leading to zero integral.

$\underline{k = n+1}$: $\displaystyle\frac{1}{(n+1)!} \int \underbrace{y h_n(y)}_{\substack{\text{monic poly. of deg } n+1 = \\ = \sum_{m=0}^{n+1} a_m h_m \text{ with } a_{n+1} = 1}} h_{n+1}(y) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = 1$

$\underline{k = n-1}$: $\displaystyle\frac{1}{(n-1)!} \int h_n(y) \cdot \underbrace{y \cdot h_{n-1}(y)}_{\text{monic poly of degree } n} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy = \frac{n!}{(n-1)!} = n$

Property (3):

By induction on n:

For $n=1$: $\quad 1 = \frac{x-y}{x-y}$ immediate.

For larger n: $\quad LHS\,(x-y) = (x-y)\sum_{k=0}^{n-1} \frac{h_k(x)h_k(y)}{k!} =$

$$= \frac{h_{n-1}(x)h_{n-2}(y) - h_{n-2}(x)h_{n-1}(y)}{(n-2)!} + \underbrace{(x-y)\frac{h_{n-1}(x)\,h_{n-1}(y)}{(n-1)!}}_{\text{apply three term recurrance}}$$

As an immediate corollary:

1) $\int \psi_k(x)\psi_\ell(x)\,dx = \delta_{k,\ell}$

2) (3-term recurrence): $x\psi_n(x) = \sqrt{n+1}\,\psi_{n+1}(x) + \sqrt{n}\,\psi_{n-1}(x)$

3) Christoffel-Darboux: $\sum_{k=0}^{n-1}\psi_k(x)\psi_k(y) = \sqrt{n}\,\frac{(\psi_n(x)\psi_{n-1}(y) - \psi_{n-1}(x)\psi_n(y))}{x-y}$

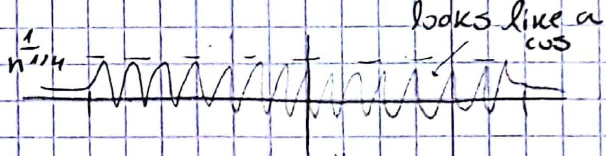
4) $\psi_n'(x) = -\frac{x}{2}\psi_n(x) + \sqrt{n}\,\psi_{n-1}(x)$

Convenient def:

$$\overline{\psi_r}(x) := n^{1/4}\,\psi_r\left(\frac{x}{\sqrt{n}}\right)$$

where it is understood than $|n-r|$ is a fixed number.

Lemma:



For fixed $|n-r|$,

$$\lim_{n\to\infty} \left|\overline{\psi_r}(t) - \frac{1}{\sqrt{\pi}}\cos\left(t - \frac{\pi r}{2}\right)\right| = 0$$

uniformly for t in a given bounded interval.

Proof of main Lemma:

$$S^N(x,y) = \frac{1}{\sqrt{N}}K^{(N)}\left(\frac{x}{\sqrt{N}}, \frac{y}{\sqrt{N}}\right) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\psi_k\left(\frac{x}{\sqrt{N}}\right)\psi_k\left(\frac{y}{\sqrt{N}}\right) \overset{CD}{=}$$

$$= \frac{\sqrt{N}\left(\psi_N\left(\frac{x}{\sqrt{N}}\right)\psi_{N-1}\left(\frac{y}{\sqrt{N}}\right) - \psi_{N-1}\left(\frac{x}{\sqrt{N}}\right)\psi_N\left(\frac{y}{\sqrt{N}}\right)\right)}{x-y}$$

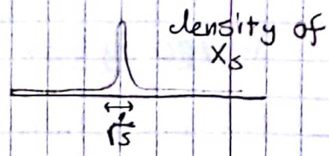
The previous lemma shows that for any $x \neq y$ this converges to $S(x,y)$.

To get rid of the $x-y$ factor, can express this using derivative

use $\frac{f(x)g(y) - f(y)g(x)}{x-y} = g(y)\int_0^1 f'(tx+(1-t)y)\,dt - f(y)\int_0^1 g'(tx+(1-t$ ... dt

this allows the uniform conv. using the formula for ψ_n'.

Laplace method (Saddle Point method):



Idea:

Suppose $\alpha: \mathbb{R} \to \mathbb{R}$ is bdd. continuous. Let $X_s \sim N(0, \frac{1}{s})$

$$\mathbb{E}[\alpha(X_s)] \xrightarrow[s\to\infty]{} \alpha(0) \quad \text{e.g. by Chebyshev}$$

$$\int_{-\infty}^{\infty} \alpha(x) \frac{\sqrt{s}}{\sqrt{2\pi}} e^{-\frac{sx^2}{2}} dx$$

$$\Downarrow$$

$$\int_{-\infty}^{\infty} \alpha(x) e^{-\frac{sx^2}{2}} dx \sim \frac{\alpha(0)}{\sqrt{s}} \sqrt{2\pi}$$

More generally, let $\gamma(x)$ be smooth & have a maximum at a.

As $s \to \infty$: $\displaystyle\int_{-\infty}^{\infty} \alpha(x) e^{s\gamma(x)} dx = \int_{-\infty}^{\infty} \alpha(x) e^{s\left(\gamma(a) - \frac{\gamma''(a)}{2}(x-a)^2 + \dots\right)} dx =$

$$= e^{s\gamma(a)} \int_{-\infty}^{\infty} \alpha(x) e^{-\frac{\gamma''(a)}{2} s \cdot (x-a)^2 + \dots} dx \sim \frac{e^{s\gamma(a)} \cdot \alpha(a)}{\sqrt{-s\gamma''(a)}} \cdot \sqrt{2\pi}$$

More formally:

We will find asymptotics for $\int_{-\infty}^{\infty} f(x)^s \alpha(x) dx$ for good f, α.

Given f, a, s_0, k, L, M, the class of α we'll look at is:

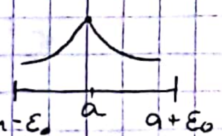
1) $|\alpha(a)| \leq k$

2) $\displaystyle\sup_{x:|x-a|<\varepsilon_0} \left| \frac{\alpha(x) - \alpha(a)}{x-a} \right| \leq L$

3) $\int f(x)^{s_0} |\alpha(x)| dx \leq M$.

Thm. (Laplace):

Let $f: \mathbb{R} \to [0, \infty)$ s.t. for some a, ε_0, c:



1) f is unimodal in the ε_0 neighborhood of a.

2) For all $\varepsilon < \varepsilon_0$, $\displaystyle\sup_{|x-a|>\varepsilon} f(x) \leq f(a) - c\varepsilon^2$. Also $f(x) > 0$ for $|x-a| < 2\varepsilon_0$.

3) f is twice cont. diff. in $(a - 2\varepsilon_0, a + 2\varepsilon_0)$

4) $f''(a) < 0$

Then, uniformly in α in the above class:

$$\int f(x)^s \alpha(x) dx \sim \sqrt{s}\, f(a)^s \sqrt{-\frac{2\pi f(a)}{f''(a)}}\, \alpha(a)$$

Proof:

$\varepsilon(s) := \varepsilon_0 \left(\frac{s_0}{s}\right)^{1/4}$ so that $\varepsilon(s) \le \varepsilon_0$ for $s \ge s_0$,

$\varepsilon(s) \longrightarrow 0$, $\sqrt{s}\,\varepsilon(s) \xrightarrow{s\to\infty} \infty$

$$\int f(x)^s \alpha(x)\,dx = \underbrace{\alpha(a) \int_{|x-a|\le\varepsilon(s)} f(x)^s\,dx}_{I} + \underbrace{\int_{|x-a|\le\varepsilon(s)} f(x)^s (\alpha(x)-\alpha(a))\,dx}_{II} + \underbrace{\int_{|x-a|>\varepsilon(s)} f(x)^s \alpha(x)\,dx}_{III}$$

For I: Write in $|x-a| \le 2\varepsilon_0$ $f(x) = f(a) e^{h(x-a)(x-a)^2}$

for $h(t) = \int_0^1 (1-r)(\log f)''(a+rt)\,dr$ using int. by parts.

Our assumptions on f give $h(0) = \frac{f''(a)}{2f(a)} < 0$ &

$e^{h\left(\frac{t}{\sqrt{s}}\right)t^2} \le e^{-\frac{ct^2}{f(a)}}$

Then: $I = f(a)^s \int_{|x-a|<\varepsilon(s)} e^{s\,h(x-a)(x-a)^2}\,dx \underset{y=\sqrt{s}(x-a)}{=} \frac{1}{\sqrt{s}} f(a)^s \int_{|y|\le\sqrt{s}\,\varepsilon(s)} e^{h\left(\frac{y}{\sqrt{s}}\right)y^2}\,dy \sim$

$\underset{\text{dominated conv.}}{\underset{s\to\infty}{\sim}} \frac{f(a)^s}{\sqrt{s}} \int_{-\infty}^{\infty} e^{h(0)y^2}\,dy$

$$|II| \le L \int_{|x-a|\le\varepsilon(s)} f(x)^s |x-a|\,dx \le L\,\varepsilon(s) \int_{|x-a|\le\varepsilon(s)} f(x)^s\,dx = L\,\varepsilon(s)\cdot I << I$$

$$|III| \le \int_{|x-a|>\varepsilon(s)} f(x)^s |\alpha(x)|\,dx \le \sup_{|x-a|>\varepsilon(s)} f(x)^{s-s_0} \cdot M \le f(a)^{s-s_0} \underbrace{\left(1 - \frac{c\,\varepsilon(s)^2}{f(a)}\right)^{s-s_0}}_{\le e^{-\frac{c\,\varepsilon(s)^2}{f(a)}(s-s_0)} \,\lesssim\, e^{-c'\sqrt{s}}} M$$

$<< 1$