Lecture 6: 19/3/2018

Prop 1:

Assume every component of ∂S contains at most 1 point from V. Then $A(S,V)$ is contractible.

Prop 2:

If V' obtained from V by adding a point $v' \in V'$ to a component of ∂S that already contain at least one point, then $A(S,V') \cong \underbrace{\Sigma}_{\text{suspension}} A(S,V)$

$\Downarrow$

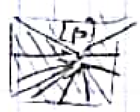
Harer's thm.

$\left(X \text{ contractible} \Rightarrow \Sigma X \text{ contr.}, \ \Sigma S^n \cong S^{n+1} \right)$

Proof of prop 1:

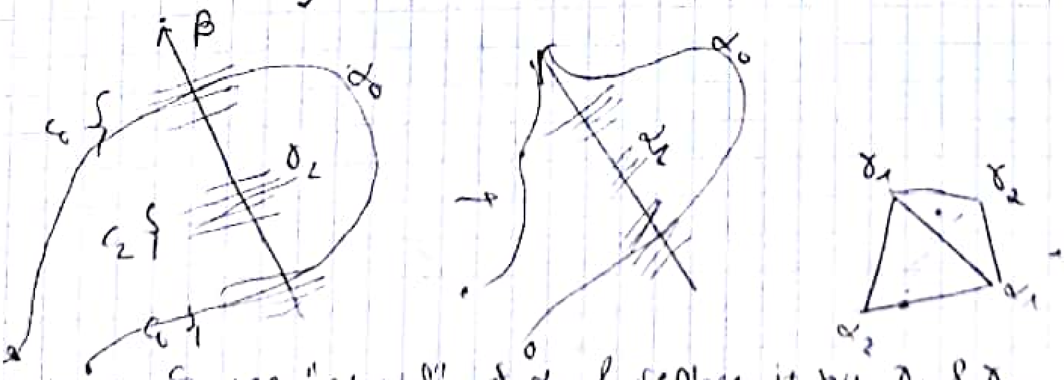
A point $p \in A(S,V)$ belongs to some simplex $p \in [\alpha_0, \dots, \alpha_k]$ & then $p = \sum_{i=0}^{k} c_i \alpha_i$, $\sum c_i = 1$, $c_i \geq 0$.

We'll construct a deformation retract from $A(S,V)$ to the Star of some vertex $[\beta]$



This is enough, as a Star is contractible.

We chose some orientation to β. If $\alpha_0, \dots, \alpha_k$ don't intersect β we do nothing; $p_t := p \ \forall t \in [0,1]$ (since p is already in the Star)



At time $\frac{\varepsilon_0}{\theta}$ we get rid of α_0 & replace it by γ_1 & γ_2.

At ~~first~~ time $\varepsilon \leq \varepsilon_0$, we replace an ε of the weight of α_0 with $\frac{\varepsilon}{2}\gamma_1 + \frac{\varepsilon}{2}\gamma_2$

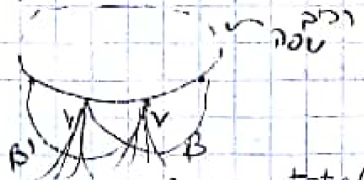
The image of ρ at time t is denoted ρ_t.

Note:

(*) $\rho_1 \in Star([\beta])$ because there are no intersections left with β.

(*) the map $\rho \mapsto \rho_t$ is continuous inside every simplex $(\forall\, t \in [0,1))$

(*) The restriction of the map $\rho \mapsto \rho_t$ to a face f of a simplex, is equal to the map on this face f.



Proof of Prop 2:

$V' = V \cup \{v'\}$

$A(S, V')$:

$\theta'(\rho)$ $\quad\theta(\rho) =$ total weight of arcs emanating from v

$X = \{ E \subseteq A(S, V') \mid \beta, \beta' \notin E \}$

$Star([\beta]) \cup star([\beta']) \cup X = A(S, V')$

$X \cap Star([\beta]) = link(\beta)$

$X \cap Star([\beta']) = link(\beta')$

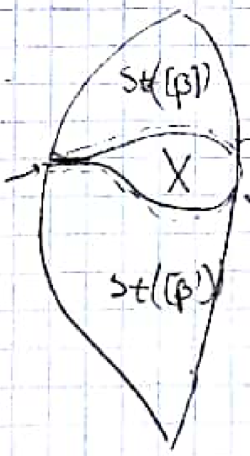
$star([\beta]) \cap star([\beta']) = link([\beta']) \cap link([\beta]) \subseteq X$

$\rho \in Star([\beta]) \iff \theta(\rho) = 0$

$\rho \in Star([\beta']) \iff \theta'(\rho) = 0$.

$\theta(\rho) = \theta(\rho') = 0$

$link(\beta) \cap link(\beta')$



$link([\beta])$

$link([\beta'])$

$\underline{\Sigma A(S,V)} = A(S,V) \times [-1,1] / \begin{smallmatrix}(x_1,1)\sim(x_2,1)\\(x_1,-1)\sim(x_2,-1)\end{smallmatrix}$

$f: X \longrightarrow A(S,V)$

$\quad p \longmapsto \bar{p}$

f defined by collapsing the segment between v & v' to a point & deleting/merging arcs if necessary.

The following is an embedding of X into $\Sigma A(S,V)$

$p \xrightarrow{\varphi} \left(\bar{p}, \underbrace{\frac{\theta(p) - \theta'(p)}{2}}_{\in[-\frac{1}{2},\frac{1}{2}]}\right)$, $\theta(p) \in [0,1]$, $\theta'(p) \in (0,1)$

(*) This function is continuous

(*) f is onto (simply take a point in $A(S,V)$ & consider it as a pt in $A(S,V')$)

$\text{Image}(\varphi) = \left\{ (p,r) \;\middle|\; r \in \left[-\frac{\tau(p)}{2}, \frac{\tau(p)}{2}\right] \right\}$

$\Sigma A(S,V)$

So we know exactly were to split the arc.

$\tau(p) = \theta + \theta'$, $r = \frac{\theta - \theta'}{2}$ so

example:

Surfaces & Commutator Length

Def:

For a group G, the commutator subgroup is

$$[G,G] = \langle \{ [u,v] \mid u,v \in G \} \rangle$$

Where $[u,v] = uvu^{-1}v^{-1}$

$G = F_r = \langle x_1, \dots, x_r \rangle$ the free group on r letters.

Lemma:

$\omega \in [F_r, F_r] \iff \omega$ is balanced. namely, for every generator x_i the number of occurrences of x_i is the same as of x_i^{-1}.

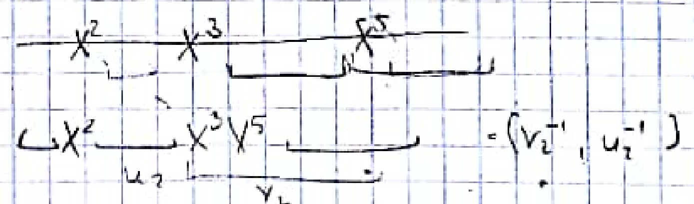
Proof:

$\Longrightarrow$: this is certainly true for a non-reduced product of non-reduced commutators, & being balanced is preserved under reduction steps.

$\Longleftarrow$:

example: $xy^2x^{-3}y^5x^2y^{-7} = xy^2x^{-3}x^2y^{-7}y^5[y^{-5},(x^2y^{-7})^{-1}]$

(*) $[a,b] = ([b,a])^{-1}$



⋮

Def:

$$c.l.(\omega) = \min \{ g \mid \omega = [u_1,v_1] \cdots [u_g,v_g] \}$$

commutator length.

Q: How can one find $c.l.(\omega)$?